Mathematical Statistics - Section 1 - NYU Spring 2019 - Homework 1

Questions preceded by a (*) are either more technical or require more initiative.

We take for statistical model the family of exponential distributions. Here $\Theta = (0, +\infty)$ and for $\theta \in \Theta$, the exponential distribution with parameter θ is given by the probability density

$$P_\theta(x) := \theta e^{-\theta x} \text{ on } [0, +\infty).$$

1. For any θ, show that the mean $m(\theta) := \int_0^{+\infty} x P_\theta(x) dx$ is finite, and compute it.

2. Show that the variance $V(\theta) := \int_0^{+\infty} (x - m(\theta))^2 P_\theta(x) dx$ is finite, and compute it.

3. Let $X_1, \dots, X_n$ be an observation of P_θ (i.e. $X_1, \dots, X_n$ are *independent* and *identically distributed* with distribution P_θ). Let $\hat{m}$ be the empirical mean, defined as

   $$\hat{m} := \frac{1}{n} \sum_{i=1}^{n} X_i.$$

   Compute the expectation and the variance of $\hat{m}$. Recall why $\hat{m}$ is a consistent estimator of $m(\theta)$.

4. We recall a useful inequality, known as the Bienaymé–Chebyshev inequality: if Y is a random variable such that $E[Y] = 0$ and $E[Y^2]$ is finite, then

   $$P(Y \geq t) \leq \frac{1}{t^2} E[Y^2].$$

   Use this inequality to prove that, for any $\varepsilon > 0$, we have

   $$P\left(|\hat{m} - m(\theta)| \geq \varepsilon\right) \leq \frac{2V(\theta)}{\varepsilon^2 n}.$$

   Explain why it gives a *quantitative* estimate on the consistency of $\hat{m}$.

5. We define a quantity $\hat{V}$ (the empirical variance) as

   $$\hat{V} := \frac{1}{n} \sum_{i=1}^{n} (X_i - \hat{m})^2.$$

   Compute the expectation of $\hat{V}$.

6. Is $\hat{V}$ an unbiased estimator of $V(\theta)$?

7. Is $\hat{V}$ an *asymptotically* unbiased estimator of $V(\theta)$?

8. (*) Is $\hat{V}$ a *consistent* estimator of $V(\theta)$?