Stats - HW1 - Solution

1) and 2) : integration by parts, obtain

$$m(\theta) = \frac{1}{\theta} \quad ; \quad V(\theta) = \frac{1}{\theta^2}$$

3)

- By linearity of the expectation, we have

$$\mathbb{E}[\hat{m}] = \mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}[X_i] = \frac{1}{n}\cdot n \cdot m(\theta) = m(\theta)$$

- Since the X_i's are independent, we have

$$\mathrm{Var}(\hat{m}) = \mathrm{Var}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2}\sum_{i=1}^{n}\mathrm{Var}(X_i) = \frac{1}{n^2}\cdot n \cdot V(\theta) = \frac{V(\theta)}{n}.$$

- The consistency of $\hat{m}$ follows from the law of large numbers.

4) We can write

$$P(|\hat{m} - m(\theta)| \geq \varepsilon) = P\big((\hat{m} - m(\theta) \geq \varepsilon) \cup (\hat{m} - m(\theta) \leq -\varepsilon)\big)$$
$$\leq P(\hat{m} - m(\theta) \geq \varepsilon) + P(\hat{m} - m(\theta) \leq -\varepsilon)$$

Applying Bienaymé - Chebyshev inequality twice, we obtain

$$P(|\hat{m} - m(\theta)| \geq \varepsilon) \leq 2 \cdot \frac{V(\hat{m})}{\varepsilon^2} = \frac{2 \cdot V(\theta)}{\varepsilon^2 n}.$$

Rem. The question and the result are correct, but not stated in an optimal fashion.

a) We do not need $\mathbb{E}[Y] = 0$ in the assumption of the inequality

b) We have in fact $P(|Y| \geq t) \leq \frac{1}{t^2}\mathbb{E}[Y^2]$ with an absolute value

So the sharper inequality $P(|\hat{m} - m(\theta)| \geq \varepsilon) \leq \frac{V(\theta)}{\varepsilon^2 n}$ is also true.

5. By linearity of the expectation, we have

$$\mathbb{E}[\hat{V}] = \mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n}(X_i - \hat{m})^2\right]$$

$$= \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\left[(X_i - \hat{m})^2\right]$$

The variables $(X_i - \hat{m})^2$ are identically distributed, so

$$\mathbb{E}[\hat{V}] = \frac{1}{n}\cdot n \cdot \mathbb{E}\left[(X_1 - \hat{m})^2\right] = \mathbb{E}\left[(X_1 - \hat{m})^2\right].$$

To compute it, we write

$$(X_1 - \hat{m})^2 = X_1^2 + \hat{m}^2 - 2\, X_1 \cdot \hat{m}$$

We have $\mathbb{E}[X_1^2] = \mathbb{E}[X_1]^2 + \mathrm{Var}(X_1)$

$$= m(\theta)^2 + V(\theta)$$

$$\mathbb{E}[\hat{m}^2] = \mathbb{E}[\hat{m}]^2 + \mathrm{Var}(\hat{m})$$

$$= m(\theta)^2 + \frac{V(\theta)}{n} \quad (\text{by Q. 3})$$

$X_1 \cdot \hat{m} = X_1 \cdot \left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)$, and the X_i's are independent

$$\mathbb{E}[X_1 \cdot \hat{m}] = \frac{1}{n}\mathbb{E}[X_1^2] + \frac{1}{n}\sum_{i=2}^{n}\mathbb{E}[X_1 X_i]$$

$$= \frac{1}{n}\left(m(\theta)^2 + V(\theta)\right) + \frac{n-1}{n}\mathbb{E}[X_1]\,\mathbb{E}[X_i]$$

$$= \frac{1}{n}\left(m(\theta)^2 + V(\theta)\right) + \frac{n-1}{n} m^2(\theta)$$

$$= \frac{V(\theta)}{n} + m^2(\theta)$$

So

$$\mathbb{E}[\hat{V}] = m(\theta)^2 + V(\theta) + m(\theta)^2 + \frac{V(\theta)}{n} - 2\,\frac{V(\theta)}{n} - 2\,m(\theta)^2$$

$$\boxed{= V(\theta)\left(1 - \frac{1}{n}\right)}$$

6. No, because $\mathbb{E}[\hat{V}] \neq V(\theta)$

7. Yes, because $\lim_{n\to\infty} \mathbb{E}[\hat{V}] = \lim_{n\to\infty} \left(1 - \frac{1}{n}\right) V(\theta)$

$$= V(\theta)$$

8. let us write

$$\hat{V} = \frac{1}{n}\sum_{i=1}^{n}(X_i - \hat{m})^2$$

$$= \frac{1}{n}\sum_{i=1}^{n}\left(X_i - m(\theta) + m(\theta) - \hat{m}\right)^2$$

$$= \frac{1}{n}\sum_{i=1}^{n}\left[(X_i - m(\theta))^2 + (m(\theta) - \hat{m})^2 + 2(X_i - m(\theta))(m(\theta) - \hat{m})\right].$$

a) The quantity $\frac{1}{n}\sum_{i=1}^{n}(X_i - m(\theta))^2 \xrightarrow[n\to\infty]{\mathbb{P}} \mathbb{E}[(X - m(\theta))^2] = V(\theta)$ by the law of large numbers applied to the r.v. $(X_i - m(\theta))^2$

b) $\frac{1}{n}\sum_{i=1}^{n}(m(\theta) - \hat{m})^2 = (m(\theta) - \hat{m})^2$ and we know that $\hat{m}$ is a consistent estimator of $m(\theta)$, so $\hat{m} \xrightarrow[n\to\infty]{\mathbb{P}} m(\theta)$, which implies $(m(\theta) - \hat{m})^2 \xrightarrow[n\to\infty]{\mathbb{P}} 0$.

c) We have $\frac{1}{n}\sum_{i=1}^{n} 2(X_i - m(\theta)) \cdot (m(\theta) - \hat{m})$

$$= 2(m(\theta) - \hat{m})\left[\frac{1}{n}\sum_{i=1}^{n}(X_i - m(\theta))\right]$$

$$= -2(\hat{m} - m(\theta))^2 \xrightarrow[n\to\infty]{\mathbb{P}} 0$$ by the same argument as above.

So $\hat{V} \xrightarrow[n\to\infty]{\mathbb{P}} V(\theta)$, it is a consistent estimator.