Math Stats – HW4 – Solution

1. Let $X_1, \ldots, X_n$ be an observation, the likelihood reads

$$\mathcal{L}_n(\lambda) = \prod_{i=1}^{n} \mathbb{P}_\lambda(X_i)$$

$$= \prod_{i=1}^{n} e^{-\lambda} \frac{\lambda^{X_i}}{(X_i)!}$$

Taking the log, we obtain

$$\log \mathcal{L}_n(\lambda) = \sum_{i=1}^{n} -\lambda + X_i \log\lambda - \log((X_i)!)$$

$$= -n\lambda + \log\lambda \left(\sum_{i=1}^{n} X_i\right) - \underbrace{\sum_{i=1}^{n} \log(X_i!)}_{\text{does not depend on } \lambda.}$$

We take the derivative and find the extremal points of $\log \mathcal{L}_n$.

$$\frac{d}{d\lambda} \log \mathcal{L}_n(\lambda) = -n + \frac{\sum_{i=1}^{n} X_i}{\lambda} = 0 \text{ ? for } \lambda = \frac{\sum_{i=1}^{n} X_i}{n}$$

It is easy to check that $\frac{\sum_{i=1}^{n} X_i}{n}$ is indeed the point at which $\mathcal{L}_n$ is maximal, and thus the MLE $\hat{\lambda}_n$ is given by $\boxed{\frac{1}{n}\sum_{i=1}^{n} X_i}$.

2. First moment

$$\mathbb{E}_{U_\theta}[X] = \int_0^\theta x \frac{dx}{\theta} = \boxed{\frac{\theta}{2}}$$

Second moment

$$\mathbb{E}_{U_\theta}[X^2] = \int_0^\theta x^2 \frac{dx}{\theta} = \boxed{\frac{\theta^2}{3}}$$

Let $X_1, \ldots, X_n$ be an observation. A possible estimator based on the method of moments would be:

$$\boxed{\begin{aligned} \hat{\theta}_{n,1} &= \frac{2}{n}\sum_{i=1}^{n} X_i \quad \text{(using first moment)} \\ \hat{\theta}_{n,2} &= \sqrt{\frac{3}{n}\sum_{i=1}^{n} X_i^2} \quad \text{(using second moment)} \end{aligned}}$$

3. The estimator based on first moment is

$$\hat{\theta}_{n,1} := \frac{2}{n} \sum_{i=1}^{n} X_i$$

$$\mathrm{Var}_{U_\theta}\left[\hat{\theta}_{n,1}\right] = \frac{4}{n^2} \sum_{i=1}^{n} \mathrm{Var}(X_i) \quad \text{(because the } X_i\text{'s are independent)}$$

$$= \frac{4}{n^2} \cdot n \left(\mathbb{E}_{U_\theta}[X^2] - \mathbb{E}_{U_\theta}[X]^2 \right)$$

$$= \frac{4}{n} \left(\frac{\theta^2}{3} - \frac{\theta^2}{4} \right) \boxed{= \frac{\theta^2}{3n}}$$

4. $\mathbb{P}\left(\max X_i \in [x, x+\delta] \right)$?

$\max X_i \in [x, x+\delta] \iff$ $n-1$ points are $\leq x+\delta$ $\quad \left(\frac{x+\delta}{\theta_*}\right)^{n-1}$

1 point is in $[x, x+\delta]$ $\quad \frac{\delta}{\theta_*}$

(we don't know which one) $\quad n$

So $\mathbb{P}\left(\max X_i \in [x, x+\delta] \right) \approx \left(\frac{x+\delta}{\theta_*} \right)^{n-1} \cdot \frac{\delta}{\theta_*} \cdot n$

Dividing by δ and taking $\delta \to 0$, we get a pdf given by

$$\boxed{\frac{n \cdot x^{n-1}}{\theta_*^{\,n}}} \quad \text{for } x \in [0, \theta_*]$$

5. Expectation

$$\int_0^{\theta_*} x \cdot \frac{n x^{n-1}}{\theta_*^{\,n}} \, dx = \frac{n}{n+1} \cdot \frac{\theta_*^{\,n+1} - 0}{\theta_*^{\,n}}$$

$$= \frac{n}{n+1} \theta_*$$

6. Second moment

$$\int_0^{\theta_*} x^2 \cdot \frac{n x^{n-1}}{\theta_*^{\,n}} \, dx = \frac{n}{n+2} \theta_*^2, \quad \text{So Variance} = \left[\frac{n}{n+2} - \left(\frac{n}{n+1} \right)^2 \right] \theta_*^2$$

$$\text{Variance} = \frac{n}{(n+2)(n+1)^2} \theta_*^2 \underset{n \to \infty}{\approx} \frac{1}{n^2} \theta_*^2$$ much smaller than in Q. 3.