Mathematical Statistics - Section 1 - NYU Spring 2019 - Homework 5

In this exercise, we will denote by M the constant

$$M := \int_{-\infty}^{+\infty} e^{-x^4} dx,$$

which we will **not** try to compute.

1. Show that, for any $\gamma > 0$, the function $x \mapsto f(x;\gamma)$ defined by

$$f(x;\gamma) := \frac{1}{M}\gamma e^{-\gamma^4 x^4},$$

is a pdf on $(-\infty, \infty)$ (*Hint: linear change of variables*)

2. Let $X_1, \dots, X_n$ be an observation, with true parameter $\gamma_\star$. Compute the max-likelihood estimator $\widehat{\gamma}_n$ of $\gamma_\star$ (*Hint: follow the usual strategy. Write the likelihood, take the log, compute the derivative etc.*)

3. Show that the Fisher information[1] of this statistical model can be written as

$$I_1(\gamma) = \frac{A}{\gamma^2},$$

for some constant A that we will not try to compute. (*Hint: use the formula seen in class for the Fisher information, and do a linear change of variables to find an expression as desired.*)

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$\star \quad \star$

We now **fix** some $\gamma > 0$, and consider the following family of pdf's, with parameter $\mu \in (-\infty, +\infty)$

$$g(x;\mu) := \frac{1}{M}\gamma e^{-\gamma^4 (x-\mu)^4}.$$

4. Let $X_1, \dots, X_n$ be an observation with true parameter $\mu_\star$. Give an equation that characterizes the max-likelihood estimator $\widehat{\mu}_n$ of $\mu_\star$, but don't try to solve it. (*Hint: same strategy as above.*)

5. Show that the Fisher information[1] of this statistical model, can be written as:

$$I_2(\mu) = B\gamma^2,$$

where B is a constant (not depending on μ, γ) that we will not try to compute. (*Hint: same strategy as above.*)

[1] We add a subscript in I_1, I_2, I_3 because we will compute several Fisher informations for several models.

6. We can readily see that:

   (a) The Fisher information does not depend on μ.

   (b) When γ is large, the Fisher information is also large.

   Why is that true, intuitively? (*Hint: think about "personality"*)

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$\star \quad \star$

Finally, we decide to return to the initial problem, but we change the parametrization. Here we let $\sigma > 0$ be the parameter, and we consider the family of pdf's given by

$$h(x;\sigma) := \frac{1}{M}\frac{1}{\sigma}e^{-x^4/\sigma^4},$$

where M is the same constant as in the beginning. Observe that

$$h(x;\sigma) = f(x;\frac{1}{\gamma}),$$

where f was defined above - so we are considering the *same* family of pdf's, but parametrized differently, the correspondence between parameters being given by $\sigma \leftrightarrow \frac{1}{\gamma}$.

7. Let $X_1, \ldots, X_n$ be an observation with real parameter $\sigma_\star$. Compute the max-likelihood $\widehat{\sigma}_n$ of $\sigma_\star$. (*Hint: same as usual*).

8. Assume that $\sigma_\star, \gamma_\star$ are connected by the relation $\sigma_\star = \frac{1}{\gamma_\star}$, observe that

$$\widehat{\sigma}_n = \frac{1}{\widehat{\gamma}_n},$$

   where $\widehat{\gamma}_n$ is the MLE for $\gamma_\star$ found in question 2. Of which property of max-likelihood estimators is this an illustration?

9. Show that the Fisher information, in this case, can be written as:

$$I_3(\sigma) = \frac{C}{\sigma^2},$$

   for a constant C that we will not try to compute. (*Hint: same as above.*)

10. (Optional) Observe that, if $\sigma = \frac{1}{\gamma}$, we do not have

$$I_3(\sigma) = \frac{1}{I_1(\gamma)},$$

   so the Fisher information does not enjoy the same nice property as the MLE. Based on the asymptotic normality result for MLE, can you make up a rule of transformation of the Fisher information under re-parametrization of the family?