1) It is non-negative, and we can check that

$$\int_{-\infty}^{+\infty} f(x;\gamma)\,dx = \int_{-\infty}^{+\infty} \frac{1}{M}\gamma e^{-\gamma^4 x^4}\,dx$$

$$\overset{(u=\gamma x}{=} \int_{-\infty}^{+\infty} \frac{1}{M} e^{-u^4}\,du = \frac{M}{M} = 1 \text{, so } f(\cdot\,;\gamma) \text{ is a pdf.}$$

2) likelihood function

$$\mathcal{L}_n(\gamma) = \prod_{i=1}^{n} \frac{1}{M}\gamma e^{-X_i\gamma^4}$$

Log-likelihood

$$\log \mathcal{L}_n(\gamma) = -\cancel{n}\log M + n\log\gamma - \sum_{i=1}^{n} X_i\gamma^4$$

Derivative $=0$?

$$\frac{n}{\gamma} - 4\sum_{i=1}^{n}\gamma^3 X_i = 0 \iff \gamma^4 = \frac{n}{4\sum_{i=1}^{n} X_i}$$

So MLE $\boxed{\hat{\gamma}_n := \left(\frac{1}{\frac{4}{n}\sum_{i=1}^{n} X_i}\right)^{\frac{1}{4}}}$

3) Fisher information

$$I_1(\gamma) = -\int_{-\infty}^{+\infty} \frac{\partial^2 \log f(x;\gamma)}{\partial\gamma^2} f(x;\gamma)\,dx$$

Compute first: $\log f(x;\gamma) = -\log M + \log\gamma - \gamma^4 x^4$

$$\frac{\partial}{\partial\gamma}\ \text{———} = \frac{1}{\gamma} - 4\gamma^3 x^4$$

$$\frac{\partial^2}{\partial\gamma^2}\ \text{———} = -\frac{1}{\gamma^2} - 12\gamma^2 x^4$$

So $I_1(\gamma) = \int_{-\infty} \left(\frac{1}{\gamma^2} + 12\gamma^2 x^4\right) \frac{\gamma}{M} e^{-\gamma^4 x^4} \, dx$

$(u = \gamma x)$ change of variables

$$= \int_{-\infty}^{+\infty} \left(\frac{1}{\gamma^2} + \frac{12\, u^4}{\gamma^2}\right) \frac{1}{M} e^{-u^4} \, du$$

$$= \frac{1}{\gamma^2} \underbrace{\left(1 + \int_{-\infty}^{+\infty} \frac{12 u^4 e^{-u^4} du}{M}\right)}$$

Some constant A

4) Likelihood $\mathcal{L}_n(\mu) = \frac{1}{M^n} \gamma^n e^{-\gamma^4 \sum_{i=1}^{n} (X_i - \mu)^4}$

Log-likelihood $\log \mathcal{L}_n(\mu) = -n \log M + n \log \gamma - \gamma^4 \sum_{i=1}^{n} (X_i - \mu)^4$

Derivative $= 0$? $\quad -\gamma^4 \cdot 4 \sum_{i=1}^{n} (X_i - \mu)^3 = 0$

So $\boxed{\sum_{i=1}^{n} (X_i - \mu)^3 = 0}$ is the "MLE equation", but not easy to solve analytically.

5) Same as above

$$I_2(\mu) = -\int_{-\infty}^{+\infty} \frac{\partial^2 \log g(x;\mu)}{\partial \mu^2} g(x;\mu) \, dx$$

$\log(x,\mu) = -\log M + \log \gamma - \gamma^4 (x-\mu)^4$

$\frac{\partial}{\partial \mu} \quad = +4\gamma^4 (x-\mu)^3$

$\frac{\partial^2}{\partial \mu^2} \quad = -12\gamma^4 (x-\mu)^2$

So $I_2(\mu) = \int_{-\infty}^{+\infty} 12\gamma^4 (x-\mu)^2 \gamma \frac{e^{-(x-\mu)^4 \gamma^4}}{M} dx$

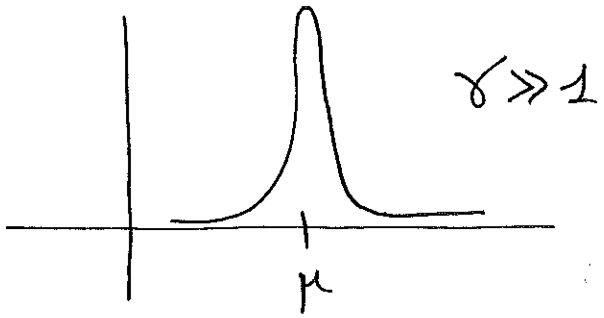
$(u = \gamma(x-\mu))$

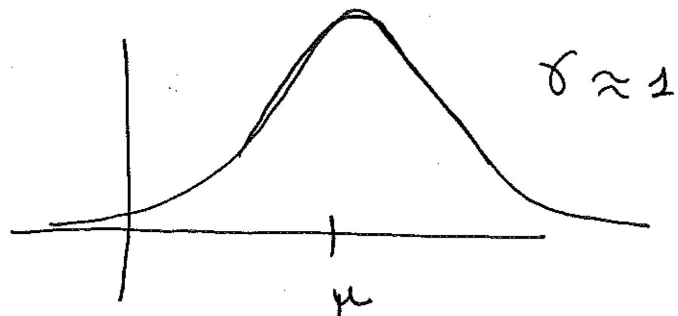
$$= 12\gamma^2 \int_{-\infty}^{+\infty} \frac{u^2 e^{-u^4}}{M} du$$

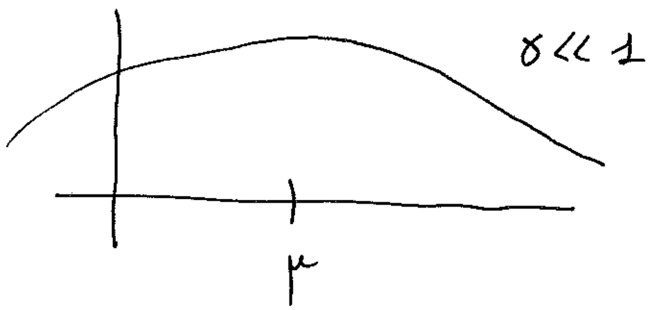
$\boxed{= \gamma^2 \cdot B}$

where $B = \frac{12}{M} \int_{-\infty}^{+\infty} e^{-u^4} u^2$ some constant.

6] b) is the clearest one.







when γ is large, the pdf is sharply peaked around μ,
when γ is small, the pdf is "flatter".
So γ large $\to$ more "personality"
$\to$ more information.

a) is simply the fact that the pdf for various μ are obtained by translating the graph, but they have all the same shape, hence the same amount of "personality".

7] likelihood

$$\mathcal{L}_n(\sigma) = \frac{1}{M^n}\frac{1}{\sigma^n} e^{-\frac{1}{\sigma^4}\sum_{i=1}^n x_i^4}$$

$$\log \mathcal{L}_n(\sigma) = -n\log M - n\log\sigma - \frac{1}{\sigma^4}\sum_{i=1}^n x_i^4$$

Derivative $=0$? $$-\frac{n}{\sigma} + \frac{4}{\sigma^5}\sum_{i=1}^n x_i^4 = 0$$

So $$\sigma = \left(\frac{4}{n}\sum_{i=1}^n x_i^4\right)^{1/4}$$

8] "Equivariance"

Write $\hat\sigma_n - \sigma_* \approx \sqrt{n}\,\mathcal{N}(0, 1/I_*^\sigma(\sigma_*))$
$\hat\gamma_n - \gamma_* \approx \sqrt{n}\,\mathcal{N}(0, 1/I^\gamma(\dots))$

9] Same computation as before, essentially

10] The fisher information is indeed not equivariant. Assume

$$\sigma = \varphi(\gamma)$$

is a transformation of the parameters.

$$\hat\sigma_n = \varphi(\hat\gamma_n)$$

for MLE, because MLE is equivariant.