HW6 - Solution

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a) Power function $\beta(\theta) = \mathbb{P}_\theta(Y > c)$

$= \mathbb{P}_\theta(\max\{X_1, \dots, X_n\} > c)$

$= 1 - \mathbb{P}_\theta(\max \leq c) \begin{cases} = 1 - \left(\frac{c}{\theta}\right)^n & \text{if } c \leq \theta \\ = 0 & \text{if } c \geq \theta \end{cases}$

b) We have size $= \beta\left(\frac{1}{2}\right) = \begin{cases} 1 - (2c)^n & c \leq \frac{1}{2} \\ 0 & c \geq 1/2 \end{cases}$

Size 0.05

$\iff 1 - (2c)^n = 0.05 \iff \boxed{\frac{(0.95)^{1/n}}{2} = c}$

c) $1 - (2 \times 0.48)^{20} = 1 - (0.96)^{20} \approx 1 - 0.44 \approx \boxed{0.56}$
p-value.

I would keep H_0

d) p-value $= 0$ Reject H_0 with confidence, even certainty!

Rem. This is not a great test. If $\theta = \frac{1}{2}$, the max should be around $\frac{1}{2}$, and we should reject if it is too small.

6

θ = probability of dying before rather than after.

Belief $\boxed{\theta = \frac{1}{2}}$

$n = 1919$

$k = 922$ # deaths before rather than after.

Pearson's χ^2-statistic $\quad T = \dfrac{\left(922 - \frac{1919}{2}\right)^2}{\frac{1919}{2}} + \dfrac{\left(997 - \frac{1919}{2}\right)^2}{\frac{1919}{2}}$

$$= \frac{1406.25}{959.5} + \frac{1406.25}{959.5}$$

$$= 2.93$$

should be a χ^2 – distribution

p-value $= \mathbb{P}_{\chi^2}(X \geq 2.93)$ $\boxed{\text{between } 5\% \text{ and } 10\%}$ Reject H_0

Confidence interval? Can use empirical mean.

$$\hat{\theta} = \frac{922}{1919} = 0.48 \qquad n = 1919$$

This is way too good to be true

$\simeq \frac{3}{10000}$

$$\frac{(\hat{\theta} - \theta)\sqrt{n}}{\hat{\theta}(1-\hat{\theta})} \approx \mathcal{N}(0,1)$$

Should be square root of theta(1-theta)

$$|\theta - \hat{\theta}| \leq \frac{c\,\hat{\theta}(1-\hat{\theta})}{\sqrt{n}} \quad \text{with proba} \lessapprox \mathbb{P}(|\mathcal{N}(0,1)| \geq c)$$

$$|\theta - \hat{\theta}| \leq \frac{c^4}{4 \times 959.5}$$

Should be 2 instead of 4

959.5 is (1919)/2, whereas it should be (1919)^(1/2)

$$\theta \in \left[0.48 - \frac{1.4}{4 \times 959},\ 0.48 + \frac{1.4}{4 \times 959}\right]$$

$c = 1.4$

with 95% confidence

7) a) Compute $\hat{\mu}_1$, $\hat{\mu}_2$, and $\hat{\mu}_1 - \hat{\mu}_2$ (difference of empirical means)

Then compute standard errors.

If we consider that 10 is a large number, we can perform Wald's test and check wether $\mathbb{P}\left(|\mathcal{N}(0,1)| \geq \frac{\hat{\mu}_1 - \hat{\mu}_2}{\sqrt{\frac{se_1^2}{8} + \frac{se_2^2}{10}}} \right)$ is small or not...

8) The distribution of T is $\mathcal{N}\left(\theta, \frac{1}{n}\right) = \theta + \mathcal{N}(0,1) \cdot \frac{1}{\sqrt{n}}$

a) So $\mathbb{P}_{\theta=0}(T > c) = \mathbb{P}_{\theta=0}\left(\underbrace{\theta}_{=0} + \frac{1}{\sqrt{n}} \mathcal{N}(0,1) > c \right)$

$$= \mathbb{P}\left(\mathcal{N}(0,1) > c\sqrt{n} \right)$$

if we want $\leq \alpha$, take $z(\alpha)$ s.t. $\mathbb{P}(\mathcal{N}(0,1) \geq z) \leq \alpha$

and let $c = \frac{z(\alpha)}{\sqrt{n}}$

b) $\beta(1) = \mathbb{P}_{\theta=1}\left(1 + \frac{1}{\sqrt{n}} \mathcal{N}(0,1) > c \right)$

c) as $n + \infty$, since $c \to 0$ we have indeed

$$\beta(1) \xrightarrow[n+\infty]{} \mathbb{P}_{\theta=1}(1 > 0) = 1.$$

9) By asymptotic normality of MLE,

$$\frac{\hat{\theta} - \theta_1}{\hat{se}} \longrightarrow \mathcal{N}(0,1)$$

so $\frac{\hat{\theta} - \theta_0}{\hat{se}} = \underbrace{\frac{\hat{\theta} - \theta_1}{\hat{se}}}_{\longrightarrow \mathcal{N}(0,1)} + \underbrace{\frac{\theta_1 - \theta_0}{\hat{se}}}_{\longrightarrow +\infty}$ and thus $\beta(\theta_1) \to 1$.