HW 7 - Solution

1] We have the exponential family, and we're making an hypothesis on the parameter θ.

We know (among other things) that $\mathbb{E}[X] = \frac{1}{\theta}$, $\mathbb{V}[X] = \frac{1}{\theta^2}$

The CLT ensures $\frac{\sqrt{n}}{\sqrt{V(X)}}$

$$\Big(\frac{1}{n}\sum_{i=1}^{n} X_i - \underbrace{\mathbb{E}[X]}_{1/\theta}\Big)\sqrt{n}\,\theta \xrightarrow[n\to\infty]{\text{distr.}} \mathcal{N}(0,1)$$

$\theta \geq 1 \Leftrightarrow \frac{1}{\theta} \leq 1$, so if H_0 is true

$\frac{1}{n}\sum_{i=1}^{n} X_i$ concentrates around $\frac{1}{\theta} \leq 1$,

in particular $\left(\frac{1}{n}\sum_{i=1}^{n} X_i - 1\right)\sqrt{n}$ is either very negative or ≈ 1 (of order 1)

On the other hand, if H_0 is not true

$\frac{1}{n}\sum_{i=1}^{n} X_i$ concentrates around $\frac{1}{\theta} > 1$,

so $\left(\frac{1}{n}\sum_{i=1}^{n} X_i - 1\right)\sqrt{n}$ is very positive.

$\longrightarrow$ We choose $T = \left(\frac{1}{n}\sum_{i=1}^{n} X_i - 1\right)\sqrt{n}$ as a test statistic

$R = \{T \geq c\}$ rejection region,

with c to be chosen in a minute.

<u>Size?</u> $\sup_{\theta\in H_0} \mathbb{P}_\theta(T\in R) = \underbrace{\mathbb{P}_{\theta=1}(T \geq c)}_{\leq 0.1}$

So take $c \approx 1.208$ for ≤ 0.1

<u>Note</u> We could also take $T = \left(\frac{1}{n}\sum X_i - 1\right)$ and $c = \frac{1.208}{\sqrt{n}}$

Here we have $\frac{1}{n}\sum_{i=1}^{n} X_i = \frac{10.96}{15} = 0.73$

So we should keep H_0 because $(0.73 - 1) \cdot \sqrt{15} = -1.045 < 0 \leq 1.2$

⚠! p Value is quite meaningless here, since we have kept H_0.

It would be more interesting to build a confidence interval.

2) 1) We have

$$\mathbb{E}[Z_{a,b}^2] = \mathbb{E}[Z_{a,b}]^2 + V(Z_{a,b})$$
$$= a^2 + b$$

2)
$$\begin{cases} a + \mathcal{N}(0,\lambda) = \mathcal{N}(a,\lambda) \\ \lambda \mathcal{N}(0,\mu) = \mathcal{N}(0, \lambda^2\mu) \end{cases}$$

3)
$$\mathbb{E}[c(f(x), Y)]$$
$$= \mathbb{E}[|f(x) - (d + \varepsilon \mathcal{N}(0,1))|^2]$$
$$= \mathbb{E}_x[\mathbb{E}_{Noise}[|\mathcal{N}(f(x) - d, \varepsilon^2)|^2]]$$
$$= \mathbb{E}_x[|f(x) - d|^2 + \varepsilon^2]$$
$$= \varepsilon^2 + \mathbb{E}[|f(x) - d|^2]$$

minimal when $f(x)$ is always $= d$. and then $= \varepsilon^2$

4) Average risk

$$\mathbb{E}\left[\left|\frac{1}{n}\sum_{i=1}^{n} Y_i - d - \varepsilon\mathcal{N}(0,1)\right|^2\right]$$
$$= \mathbb{E}_{Data}\left[\mathbb{E}_{Noise}\left[\mathcal{N}\left(\frac{1}{n}\sum_{i=1}^{n} Y_i - d, \varepsilon^2\right)\right]\right]$$
$$= \mathbb{E}_{Data}\left[\left|\frac{1}{n}\sum_{i=1}^{n} Y_i - d\right|^2 + \varepsilon^2\right]$$
$$= \frac{Var(Y)}{n} + \varepsilon^2$$

as $n \to \infty$