HW 8 - Solutions

1] By definition, we must have

$$\begin{cases} (a\vec{U} + b\vec{X} + c\vec{X^2} - \vec{Y}) \cdot \vec{U} = 0 \\ ((a\vec{U} + b\vec{X} + c\vec{X^2}) - \vec{Y}) \cdot \vec{X} = 0 \\ (a\vec{U} + b\vec{X} + c\vec{X^2} - \vec{Y}) \cdot \vec{X^2} = 0 \end{cases}$$

We compute

$$\vec{U} \cdot \vec{U} = n$$
$$\vec{U} \cdot \vec{X} = n\overline{X}$$
$$\vec{U} \cdot \vec{X^2} = n\overline{x^2}$$
$$\vec{X} \cdot \vec{X} = \|\vec{X}\|^2 = n\overline{X^2}$$
$$\vec{X^2} \cdot \vec{X^2} = \|\vec{X^2}\|^2 = n\overline{X^4}$$
$$\vec{X} \cdot \vec{X^2} = n\overline{X^3}$$

$$\begin{cases} an + bn\overline{X} + cn\overline{X^2} = n\overline{Y} \\ an\overline{X} + bn\overline{X^2} + cn\overline{X^3} = n\,\vec{Y} \cdot \vec{X} \\ an\overline{X^2} + bn\overline{X^3} + cn\overline{X^4} = n\,\vec{Y} \cdot \vec{X^2} \end{cases}$$

So
$$\begin{cases} a + b\overline{X} + c\overline{X^2} = \overline{Y} \\ a\overline{X} + b\overline{X^2} + c\overline{X^3} = \vec{Y} \cdot \vec{X} \\ a\overline{X^2} + b\overline{X^3} + c\overline{X^4} = \vec{Y} \cdot \vec{X^2} \end{cases}$$

2] A vector in the image of f can be written as

$$FC = \begin{pmatrix} 1 & x_1 & x_1^2 \\ & \vdots & \\ 1 & x_n & x_n^2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a + bx_1 + cx_1^2 \\ \vdots \\ a + bx_n + cx_n^2 \end{pmatrix}$$

and this equal to $a\vec{U} + b\vec{X} + c\vec{X^2}$ with $\vec{U}, \vec{X}, \vec{X^2}$ as above.

A more abstract way to find the projection of Y onto the image of F is to find C such that $Y - FC$ is $\perp$ to any FC' for $C' \in \mathbb{R}^3$,

so
$$(FC')^T (Y - FC) = 0 \quad \forall C'$$
$$(C')^T F^T (Y - FC) = 0 \quad \forall C'$$
so $F^T(Y - FC) = 0$

and thus $\boxed{C = (F^TF)^{-1} F^T Y}$