Let $C^0(\mathbb{R}, \mathbb{R})$ be the set of **all** continuous functions from $\mathbb{R}$ to $\mathbb{R}$.

Perhaps surprisingly, the following result is true:

Theorem 1. *There exists an injective (one-to-one) map ψ from $C^0(\mathbb{R}, \mathbb{R})$ to $\mathbb{R}$ itself.*

In plain words, there is a way (although an extremely cumbersome one, see below) to encode every continuous function by *a single real number*!

This abstract result points out that the distinction parametric/non-parametric is not extremely well-defined as such, because under very reasonable assumptions, **all** the models are, strictly speaking, parametric. It should be taken as a heuristic distinction: some models have natural parameters attached to them, e.g. the mean and the variance for the Gaussian family, or the coefficients of a matrix for the linear models, and some don't, e.g. the family of all probability distributions.

A possible way to make the distinction sounder is to require the parametrization $\theta \mapsto P_\theta$ to have some kind of regularity: this is possible for the "really" parametric models, but impossible for the "falsely" parametric ones.

Here is a brief sketch of how the proof of the theorem goes:

1. Let $\mathbb{Q}$ be the set of all rational numbers. It is well-known that $\mathbb{Q}$ is *countable*, i.e. there exists a bijection from $\mathbb{Q}$ to $\mathbb{N}$ (the set of natural numbers) [that might already be a bit surprising]. In other words, we may enumerate all the rational numbers as an infinite sequence $(x_1, x_2, \ldots, x_n, \ldots)$.

2. Given a function f in $C^0(\mathbb{R}, \mathbb{R})$, we may encode it by only keeping track of its values on $\mathbb{Q}$, instead of $\mathbb{R}$. This is because:

   (a) The rational numbers are dense in $\mathbb{R}$ (in other words: every real number is a limit of a sequence of rational numbers).

   (b) A continuous function is determined by its values on any dense subset of $\mathbb{R}$ (in other words: if D is a dense subset of $\mathbb{R}$ and if two continuous functions f, g are equal on D, then they must be equal on $\mathbb{R}$, so $f = g$).

3. Define a first encoding map ϕ: $C^0(\mathbb{R}, \mathbb{R}) \to \mathbb{R}^{\mathbb{N}}$ by

   $$\phi(f) := (f(x_1), f(x_2), \ldots, f(x_n), \ldots).$$

   By the arguments above, ϕ is a one-to-one map.

4. Show that there exists a one-to-one map j from $\mathbb{R}^{\mathbb{N}}$ to $\mathbb{R}$. This is quite non-trivial (and things start to be very surprising).

5. The composition $\psi := j \circ \phi$ is then a one-to-one map from $C^0(\mathbb{R}, \mathbb{R})$ to $\mathbb{R}$.