HW 2 - ODE's - Spring 2018

Due date: Tuesday, February 6^{th}

Order 1, scalar, constant coefficients Solve the following differential equations by first solving the associated homogeneous ODE and then using the "variation of the constant" method.

1. $y' + ty = t$
2. $y' + y = \cos(t)$

Order 1, system of two equations with constant coefficients We consider the following system of two first-order, scalar differential equations

$$\begin{cases} x' &= x + 3y \\ y' &= 4x + 2y. \end{cases} \tag{1}$$

1. Write this system as a differential equation of the form

$$Z' = AZ,$$

where $Z = \begin{pmatrix} x \\ y \end{pmatrix}$ and A is some 2×2 matrix.

2. Find the eigenvalues λ_1, λ_2 of A and a basis of eigenvectors. Find the associated "change of basis matrix" P such that

$$A = P\text{Diag}(\lambda_1, \lambda_2)P^{-1}.$$

3. We recall that

$$e^{tP\text{Diag}(\lambda_1,\lambda_2)P^{-1}} = P\text{Diag}\left(e^{t\lambda_1}, e^{t\lambda_2}\right)P^{-1}$$

Deduce the solutions to the system (1).

Order 2, scalar, constant coefficients Solve the following equations by computing the associated characteristic polynomial and using the general form of the solutions.

1. $y'' - y' - 2y = 0$
2. $y'' - 4y' + 4y = 0$
3. $y'' - 2y' + 10y = 0$ such that $y(0) = 0$ and $y'(0) = 1$

Higher-order, scalar, constant coefficients Solve the following equations by factorizing the associated characteristic polynomial.

1. $y^{(3)} - 2y'' - 5y' + 6y = 0$
2. $y^{(5)} - 4y^{(4)} + 4y^{(3)} = 0$

Order 1, 2×2 system (again, but not homogeneous) Using the solutions found for (1), solve

$$\begin{cases} x' &= x + 3y - 5t \\ y' &= 4x + 2y - 4. \end{cases} \tag{2}$$

You may apply the "variation of the constant" method.

Particular solutions Solve the following equations

1. $y'' - 2y' + y = e^t$
2. $y'' + y' = t$
3. $y'' - 3y' - 4y = t + e^{-t}$

For the last one, observe that the right-hand side is the sum of two terms with a "particular form" and use the superposition of solutions.

Fundamental system of solutions [Optional] Let $Y' = A(t)Y$ be an homogeneous linear ODE in dimension N (Y is a vector in $\mathbb{R}^N$, where A is defined on some interval I. Let $Y_1, dots, Y_N$ be N solutions of the equation defined on I. Prove that the following statements are equivalent:

- For all t in I, the family $(Y_1(t), \ldots, Y_N(t))$ is linearly independent in $\mathbb{R}^N$.
- There exists t_0 in I such that the family $(Y_1(t_0), \ldots, Y_N(t_0))$ is linearly independent in $\mathbb{R}^N$.