HW 3 - ODE's - Spring 2018

Due date: Tuesday, February 13^{th}

First-order linear ODE's Solve the following ODE's with given initial condition (or "Cauchy problem"). Specify the interval where the (maximal) solution is defined.

1. $$y' = \frac{1}{1+t^2}y + (1+\cos(t))e^{\tan^{-1}(t)}$$
   with $y(0) = 1$.

2. $$y' = \sqrt{t}y + t^{5/2}$$
   with $y(1) = 0$.

The non diagonalizable case We consider the following ODE

$$\begin{cases} x' &= 5x - y \\ y' &= x + 3y \end{cases}$$

1. Write the system in the form $Z' = AZ$ where A is a 2×2 matrix.

2. Compute the eigenvalues of A. Can you find a basis of eigenvectors?

3. Find a basis of $\mathbb{R}^2$ in which A has the form
   $$\begin{pmatrix} 4 & 1 \\ 0 & 4 \end{pmatrix}$$
   and let P be the associated "change of basis" matrix, such that
   $$A = P \begin{pmatrix} 4 & 1 \\ 0 & 4 \end{pmatrix} P^{-1}$$

4. Let N be the matrix
   $$N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$
   Show that its exponential is given by
   $$e^{tN} = \mathrm{Id}_2 + tN$$
   where Id_2 is the 2×2 identity matrix.

5. Conclude that
   $$e^{tA} = Pe^{4t}(\mathrm{Id}_2 + tN)P^{-1}$$
   and deduce the solutions of the ODE.

Application of Cauchy-Lipschitz Justify, as concisely and convincingly as you can, that there is existence and uniqueness of maximal solutions for the following ODE's with initial conditions.

1. $y' = t^2 e^y$ with $y(0) = 2$
2. $y'' + (y')^2 \sin(ty) = 0$ with $y(0) = 1, y'(0) = 3$.

(Do not try to solve them!)

Cauchy-Lipschitz, again Study the existence/uniqueness of maximal solutions, and their expressions, in the following cases.

1. $y' = y^2$, with $y(2) = -1/2$.
2. $y' = y^2 + 1$ with $y(1) = 1$.
3. $y' = \frac{1}{y}$ with $y(0) = 1$.

In particular, try to explain why the solutions that you find are **maximal**.

Particular solutions Go back to the "particular solutions" exercise of HW2 and convince yourself that the recipes given in class indeed yield particular solutions.

[Optional] Global solutions Let $f : (a, b) \times \mathbb{R} \to \mathbb{R}$ be continuous with respect to the first variable and C^1 with respect to the other one. Moreover, assume that there exists positive functions C_1, C_2 which are integrable[1] on (a, b) and such that

$$|f(t, y)| \leq C_1(t)|y| + C_2(t) \quad \text{for all } (t, y) \text{ in } (a, b) \times \mathbb{R}.$$

Prove that maximal solutions of $y' = f(t, y)$ are global, i.e. defined on (a, b). You may use Grönwall's lemma.

[1] You can assume "bounded" if that is easier.