HW 4 - ODE's - Spring 2018

Due date: Tuesday, February 20^{th}

Application of Cauchy-Lipschitz Justify that there is existence and uniqueness of maximal solutions for the following ODE with initial condition.

$$y'' + (y')^2 \sin(ty) = 0 \text{ with } y(0) = 1, y'(0) = 3$$

(Do not try to solve!)

Cauchy-Lipschitz, again Study the existence/uniqueness of maximal solutions, and their expressions, in the following cases.

1. $y' = y^2$, with $y(2) = -1/2$.
2. $y' = y^2 + 1$ with $y(1) = 1$.
3. $y' = \frac{1}{y}$ with $y(0) = 1$.

In particular, try to explain why the solutions that you find are **maximal**.

Particular equations Find the maximal solutions for the following ODE's (make sure to check that your solutions are *maximal*).

1. $y' = t\sqrt{y}$
2. $t^2 y' - ty - y^2 = 0$

The Euler equation Euler's equation is an ODE of the form

$$t^n y^{(n)} + a_{n-1} t^{n-1} y^{(n-1)} + \cdots + a_1 t y' + a_0 y = 0. \quad (1)$$

Where $a_0, \ldots, a_{n-1}$ are constants. For simplicity, let us assume that we consider this equation for t in $(0, +\infty)$.

We want to show that changing the time variable as $t = e^u$ transforms Euler's equation into a linear ODE.

1. Let us first treat the $n = 2$ case

   $$t^2 y^2 + aty' + by = 0,$$

   where a, b are constants. We let $y(t) = z(\ln(t))$, where z is the new unknown function, and we let $u = \ln(t)$. Express y' and y'' in terms of z', z'', t, u and show that $u \mapsto z(u)$ satisfies a linear equation on $\mathbb{R}$.

2. Now, n is arbitrary, we still let $y(t) = z(\ln(t))$, where z is the new unknown function, and $u = \ln(t)$. Show (e.g. by induction on k) that for any $k \geq 1$, there exists constants $c_2, \dots, c_k$ such that

$$y^{(k)}(t) = \frac{1}{t^k} \left(z'(\ln(t)) + c_2 z''(\ln(t)) + \cdots + c_k z^{(k)}(\ln(t)) \right).$$

3. Deduce that $u \mapsto z(u)$ satisfies some linear equation on $\mathbb{R}$ (you do not have to write it down explicitly).

[Optional] "Lowering the order" Let y_1 be a solution of

$$y'' + p(t)y' + q(t)y = 0, \tag{2}$$

different from the trivial one. Show that, after introducing the unknown function z defined by $y = y_1 z$, the equation (2) can be reduced to a first-order linear equation in $y_1^2 z$, and deduce the solutions of (2).