HW 5 - ODE's - Spring 2018

Due date: Tuesday, February 27^{th}

Power series expansion We consider the ODE

$$y'' = e^t y^2 - (y')^2.$$

We assume that the solution with initial condition $y(0) = 0$ and $y'(0) = 1$ admits a power series expansion near $t = 0$. Give the expansion of $y(t)$ near 0 up to fourth order, i.e. with an error $O(t^5)$.

Time of existence

1. We consider the ODE
$$y' = 36 \cos\left(\sqrt{1 + y^2}\right),$$
where y is an unknown real-valued function. Show that the maximal solutions are global, i.e. defined on $\mathbb{R}$.

2. Same question for the ODE
$$y' = 36\sqrt{1 + y^2}.$$

3. Now we consider the ODE
$$y' = 36(1 + y^2)^{3/5},$$
with $y(0) = 1$. Give a lower bound on the time of existence of the maximal solution.

Conserved quantities We consider the following ODE (with x the unknown function)

$$x'' + x + x^3 = 0, \tag{1}$$

which is an autonomous, second-order scalar ODE.

1. Find a conserved quantity, i.e. find a function Q on $\mathbb{R} \times \mathbb{R}$ such that, if x is a solution to (1) defined on I, we have
$$Q(x(t), x'(t)) = \text{constant for } t \text{ in } I.$$
Hint: for these questions, it is often fruitful to multiply the ODE by x' and to integrate.

2. Show that the maximal solutions to (1) are global, i.e. they are all defined on $\mathbb{R}$.

3. We want to prove that every solution is periodic. Let x_0, x_0' be in $\mathbb{R}$ and let x be the solution to (1) defined on $\mathbb{R}$ and satisfying $x(0) = x_0$ and $x'(0) = x_0'$.

(a) Prove that if there exists $T > 0$ such that $x(T) = x_0$ and $x'(T) = x'_0$, then for all t in $\mathbb{R}$ we have $x(t+T) = x(t)$ and $x'(t+T) = x'(T)$, and thus the solution is periodic.

(b) Prove that there exists $T > 0$ such that $x(T) = x_0$ and $x'(T) = x'_0$. *Hint: you may need to use the result of question 1. Keep also in mind that x is continuous and real-valued.*

4. We now consider the equation

$$x'' + cx'x^2 + x^3 = 0,$$

where c is some constant. Are there periodic solutions?