HW 6 - ODE's - Spring 2018

Due date: Tuesday, March 6^{th}

An autonomous system We consider the system

$$\begin{cases} x'(t) = & y(t) + L\left(x(t)^2 + y(t)^2 - 1\right)x(t) \\ y'(t) = & -x(t) + L\left(x(t)^2 + y(t)^2 - 1\right)y(t), \end{cases}$$

where L is some constant.

1. Solve the ODE with initial condition $x(0) = 0, y(0) = 1/2$ when $L = 0$. Now L is supposed to be positive.

2. What are the stationary points?

3. Show that the unit disk $\{x^2 + y^2 \leq 1\}$ is invariant under the flow (in both directions of the time).

4. What is the time of existence of maximal solutions? You may distinguish three cases depending on the value of $x^2 + y^2$ at the initial time. You may also look at the ODE satisfied by the quantity $x^2 + y^2$.

5. We denote by x_L, y_L the solution of the ODE with initial condition $x(0) = 0, y(0) = 1/2$. We assume that, when L is very small, x_L, y_L admit an expansion

$$x_L(t) = x_0(t) + L\tilde{x}_0(t) + \dots, \quad y_L = y_0(t) + L\tilde{y}_0(t) + \dots,$$

where x_0, y_0 are the solutions found in the first question and the errors

$$x_L(t) - (x_0(t) + L\tilde{x}_0(t)), \quad y_L - (y_0(t) + L\tilde{y}_0(t))$$

are of order L^2. Find the unknown functions $\tilde{x}_0, \tilde{y}_0$ (your argument does not have to be rigorous).

Heteroclines trajectories Let $f : \mathbb{R}^N \to \mathbb{R}$ be a C^1 function with a finite number of critical points. We consider the following autonomous ODE

$$Y' = \nabla f(Y).$$

1. Give at least one (non-trivial) condition on f which would ensure that all the maximal solutions are global, i.e. defined on $(-\infty, +\infty)$. We will now assume that all the maximal solutions are global.

2. Show that there are no periodic orbits, except maybe the constant ones. *Hint: You can observe that some natural quantity is always monotone along the flow.*

3. **(More difficult.)** Show that the only bounded orbits are the constant ones and the ones that go from one critical point to another, called *heteroclines orbits*. More precisely, the orbit $\mathcal{O}_x$ is an *heterocline orbit* if

$$\lim_{t \to -\infty} \Phi^t(x) = x_0, \quad \lim_{t \to +\infty} \Phi^t(x) = x_1,$$

where Φ^t is the flow of the ODE and x_0, x_1 are two critical points of f.