HW 7 - ODE's - Spring 2018

Due date: Tuesday, March 27^{th}

Conjugated systems Let us consider two autonomous ODE's in $\mathbb{R}^N$

$$Y' = F(Y), \quad Z' = G(Z),$$

Let Φ, Ψ be the associated flows. We assume, for simplicity, that F, G are globally Lipschitz on $\mathbb{R}^N$, so that the flows are defined everywhere and for all t.

Let $p \geq 0$ be an integer. We say[1] that the flows are C^p-conjugated if there exists a C^p-diffeomorphism[2] $h : \mathbb{R}^N \to \mathbb{R}^N$ such that for all t in $(-\infty, \infty)$

$$\Phi^t \circ h = h \circ \Psi^t. \tag{1}$$

1. Check that it is an equivalence relation between flows (reflexive, symmetric, transitive).
2. Show that if the flows are C^0-conjugated, then h induces a bijection from the orbits of Ψ onto the orbits of Φ.
3. Show that if the flows are C^1-conjugated by the diffeomorphism h, we have for any x in $\mathbb{R}^N$
$$(Dh)(x)G(x) = (F \circ h(x)), \tag{2}$$
where $(Dh)(x)$ is the Jacobian matrix of h at x.
4. Now, we consider two linear ODE's
$$Y' = AY, \quad Z' = BY.$$
Show that if A, B are *similar* matrices then the associated flows are C^p-conjugated for every p.
5. Take A, B to be
$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$$
Using the first result of last lecture ("straightening flows"), explain why the flows associated to A and B are locally C^1-conjugated away from the real axis, in the following sense: for any point x_0 that is **not** $(0,0)$, and for t small enough, the relation (1) holds for a certain map h defined around x.

[1] This might not be standard terminology.

[2] We recall that a C^p-diffeomorphism is a bijection of class C^p, whose inverse bijection is also of class C^p.

6. Explain why they cannot be locally C^1-conjugated around $(0,0)$. You can use the identity (2) (and differentiate it) to show that if the flows associated to A, B are conjugated around $(0,0)$ then A and B must be similar.

7. Let us consider the ODE's

$$\begin{cases} x' &= 2x^2y^2 + \sin(x) \\ y' &= y^3 + 2y \end{cases}, \qquad \begin{cases} x' &= \sin(x) \\ y' &= 2\sin(y) \end{cases}$$

Using Hartman-Grobman's theorem, show that the flows of these ODE's are locally C^0-conjugated around $(0,0)$.

Taylor's method We study the scalar ODE $y' = f(t,y)$ under the assumption that the map $(t,x) \mapsto f(t,x)$ is of class C^2 on $(-\infty,\infty) \times \mathbb{R}$, and that its first and second partial derivatives are all uniformly bounded.

Say we look at the time interval $t \in [0,1]$, with a given initial condition $y(0) = y^0$.

For $N \geq 1$, we consider the following numerical scheme: $y_{0,N}$ is fixed and for $0 \leq n < N$ we define

$$y_{n+1,N} := y_{n,N} + \frac{1}{N} f\left(\frac{n}{N}, y_{n,N}\right) + \frac{1}{2}\left(\frac{1}{N}\right)^2 f'\left(\frac{n}{N}, y_{n,N}\right). \tag{3}$$

This amounts to taking a fixed step-size $\frac{1}{N}$ and replacing the first-order approximation of "Explicit Euler" by a second-order approximation.

Question: Prove the convergence of this numerical method. In other words, assuming[3] that we take $\lim_{N\to\infty} y_{0,N} = y^0$, show that the quantity

$$\max_{0 \leq n \leq N} \left| y_{n,N} - y\left(\frac{n}{N}\right) \right|$$

tends to zero as $N \to \infty$. What is the speed (order) of convergence? What if we assume f to be C^p and take a Taylor's approximation of order p in the numerical scheme (3)?

[3] You may assume that $y_{0,N}$ is **equal to** y^0 if that makes your life simpler.