HW 8 - ODE's - Spring 2018

Due date: Tuesday, April 10^{th}

One-dimensional case Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and T-periodic, we consider

$$u'(t) = f(t)u(t).$$

Find a necessary and sufficient condition on f in order for all the solutions of the ODE to be bounded.

Periodic solution for Mathieu We consider the ODE

$$u''(t) + (\alpha + \beta \cos(2t))\, u(t) = 0,$$

where α, β are real coefficients.

1. Let (α, β) be fixed and assume that the ODE admits a periodic solution u. Show that one of the following cases hold:

   (a) All the solutions are periodic.

   (b) The periodic solution is even.

   (c) The periodic solution is odd.

   Hint: consider $\bar{u}(t) = u(-t)$. Also, keep in mind that the space of solutions has dimension 2...

2. Show that for b) we must have $u'(0) = 0$ and for c) we must have $u(0) = 0$.

Harmonic oscillator with external force We study the ODE

$$u'' + \omega^2 u = v, \quad u(0) = u'(0) = 0.$$

We assume that v is T-periodic, and we recall that v can be written as a Fourier series

$$v(t) = \sum_{k \in \mathbb{Z}} a_k \exp\left(\frac{2ik\pi}{T} t\right),$$

where the $\{a_k\}_{k \in \mathbb{Z}}$ are the Fourier coefficients.

1. Find the solution of the homogeneous equation.

2. *We recall the method of "superposition of solutions": when looking for a particular solution of a linear ODE with right-hand side* $w_1 + w_2$*, we can sum the particular solutions obtained for* w_1 *and* w_2. If T is not an integer multiple of $\frac{2\pi}{\omega}$, find a particular solution in the form of a Fourier series.

3. If $T = \frac{2k\pi}{\omega}$, for some integer k, find a particular solution of

$$u'' + \omega^2 u = \exp\left(\frac{2ik\pi}{T} t\right).$$

   Hint: look back for the "particular solutions" remark in the "Linear ODE" chapter.

4. Is it periodic? This is called a *secular term*.

5. Still in this case, give the form of the solution of

$$u'' + \omega^2 u = v, \quad u(0) = u'(0) = 0.$$