Syllabus for the final exam (ODE 2018)

- Computations: solving linear scalar ODE's, scalar ODE's with separated variables / homogeneous coefficients, linear ODE's with constant coefficients, finding particular solutions (by variation of the constant or with the "guessing" strategy). Linear algebraic techniques in simple cases: diagonalizing a matrix, computing its exponential, or finding a pre-image by the exponential map.

- Cauchy-Lipschitz, notion of maximal solutions, time of existence, the blow-up in finite time criterion, how to prove bounds on the time of existence by comparing an ODE to a simpler one (e.g. with Grönwall's lemma).

- Qualitative study: notion of flow (especially for autonomous systems), orbits, conserved quantities and how to use them to sketch phase portraits. Allure of the orbits near a non-stationary point ("straightening theorem for vector fields") and near a stationary point (Hartman-Grobman).

- Numerical study: what is a numerical scheme, how do we proceed to compare a numerical solution to the real one? Note that it usually involves some "hands-on" analysis using various techniques of Calculus.

- Stability: definition, use of Liapounov functions. The special case of linear ODE's with constant, or periodic, coefficients.

Concerning the final.

- Every answer must be precisely justified, unless stated otherwise.
- Every answer must use **words**, and must take the form of one or several **full sentences**. It is good practice to underline or to box the key steps of an argument and the final result of a computation (perhaps using a different color).
- Please use a **real pen**, not a pencil. Use scratch paper for your trial-and-error process and for uncertain computations. It is, of course, OK to strike out a paragraph (*pro tip:* use a pencil for this, you may end up realizing that the answer was correct).
- It is always OK to skip a question and to admit the result of a previous question. Indicate it clearly. In general, always refer precisely to the result(s) you are using, may it be the answer to a previous question (*By question 2.4, we know that...*) or a result from class (*Since* $(t, x) \mapsto F(t, x)$ *is continuous in both variables and* C^1 *in* x*, the theorem of Cauchy-Lipschitz implies...*).

Example 1: a computational question
Question: Find the solution of the following ODE with given initial condition:

$$x' = \frac{\sin(x) + 3x}{3 + \cos(x)}(1 + t^2), \quad x(0) = \pi/4.$$

You may look for an equation defining $x(t)$ implicitly for any t, without trying to find $x(t)$ explicitly.
Answer: First, let us observe that $3 + \cos(x)$ is never 0 because $|\cos|$ is bounded by 1, so that the right-hand side of the equation is always defined.

By elementary calculus, we see that the function $x \mapsto \sin(x) + 3x$ is strictly increasing and vanishes only for $x = 0$. On the other hand, the constant function equal to 0 is a solution to the ODE. By the uniqueness part of the Cauchy-Lipschitz's theorem, we deduce that any other solution will never vanish. In particular, with the initial condition $x(0) = \pi/4$ the term $\sin(x(t)) + 3x(t)$ will always be positive and we may write

$$\frac{3 + \cos(x(t))}{\sin(x(t)) + 3x(t)} x'(t) = 1 + t^2.$$

Integrating this equality between 0 and t, we obtain

$$\int_0^t \frac{3 + \cos(x(s))}{\sin(x(s)) + 3x(s)} x'(s) ds = \int_0^t (1 + s^2) ds = t + \frac{t^3}{3},$$

and the integral in the left-hand side can be computed as

$$\int_0^t \frac{3+\cos(x(s))}{\sin(x(s))+3x(s)} x'(s)ds = \int_0^t \ln\left(\sin(x(s))+3x(s)\right)' ds$$
$$= \underline{\ln(\sin(x(t))+3x(t)) - \ln\left(\sin(\pi/4)+3\pi/4\right)}.$$

We thus have, for any t

$$\boxed{\sin(x(t))+3x(t) = \left(\sin(\pi/4)+3\pi/4\right) e^{t+\frac{t^3}{3}},}$$

which defines $x(t)$ implicitly.

Example 2: a more theoretical question

Question: Let $X \mapsto F(X)$ be a continuous, locally Lipschitz function on $\mathbb{R}^d$, let $Q : X \mapsto Q(X)$ be a continuous function on $\mathbb{R}^d$ such that

$$\forall X \in \mathbb{R}^d, \quad \|X\|^3 \leq |Q(X)|$$

We assume that Q is "periodically conserved" in the following sense: if X is a solution to the ODE $X' = F(X)$ and if X is defined at times t and $t+T$, then we have

$$Q(X(t)) = Q(X(t+T)). \tag{1}$$

Prove that if a solution to $X' = F(X)$ is defined on $[0,T]$ then it is defined for all times.

Answer: Let X be a solution to $X' = F(X)$, and let us assume that X is defined on $[0,T]$. Let us argue by contradiction and assume that the time of existence of X is finite (e.g. as t increases), let us denote it by β. Since F is defined on $\mathbb{R}^d$, the "blow-up in finite time" criterion implies that

$$\lim_{t\to\beta^-} \|X(t)\| = +\infty.$$

By assumption, we have $\|X(t)\|^3 \leq |Q(X(t))|$, hence we get

$$\underline{(*) \lim_{t\to\beta^-} |Q(X(t))| = +\infty.}$$

However, from the assumption (1), it is clear that for any t in $(0,\beta)$, we have, letting $s = t - T\left\lfloor \frac{t}{T} \right\rfloor$ (where $\lfloor x \rfloor$ denotes the integer part of x, or "floor" function)

$$Q(X(t)) = Q\left(X(s)\right).$$

But $s = t - T\left\lfloor \frac{t}{T} \right\rfloor$ is a value in $[0,T]$, and $s \mapsto Q(X(s))$ is bounded on $[0,T]$ (as the composition of two continuous functions), which yields a contradiction in view of Equation (*).