Linear ODE's with periodic coefficients

1 Examples

- $y' = \sin(t)y$, solutions $Ce^{-\cos t}$. Periodic, go to 0 as $t \to +\infty$.
- $y' = -2\sin^2(t)y$, solutions $Ce^{-t-\sin(2t)/2}$. Not periodic, go to to 0 as $t \to +\infty$.
- $y' = (1+\sin(t))y$, solutions $Ce^{t-\cos(t)}$. Not periodic, do not go to 0 as $t \to +\infty$.

2 Floquet's theorem

We consider the ODE
$$Y' = A(t)Y, \tag{2.1}$$
where $t \mapsto A(t)$ is a continuous, T-periodic map from $(-\infty, +\infty)$ to $\mathcal{M}_N(\mathbb{R})$ ($N \times N$ matrices with real coefficients).

For any t in $(-\infty, +\infty)$, we introduce the **resolvent** $R(t)$, which is nothing but the flow of the ODE, i.e. for X in $\mathbb{R}^N$, we define $R(t)X$ as the value at time t of the solution to (2.1) which is equal to X at $t = 0$.

Lemma 2.1 (Properties of the resolvent).
1. *$R(0) = \mathrm{Id}$ and $R(t)$ is always invertible.*
2. *For any t, $R(t)$ is a linear map from $\mathbb{R}^N \to \mathbb{R}^N$.*
3. *For any t, we have $R'(t) = A(t)R(t)$.*

Proof.
1. By definition, and by the fact that $R(t)R(-t) = R(0)$.
2. This follows from the linearity of the ODE.
3. Exercise.

□

Theorem 1 (Floquet). *We have, for any $t \in (-\infty, +\infty)$*
$$R(t+T) = R(t)R(T), \tag{2.2}$$
and $R(t)$ can be written as
$$R(t) = U(t)e^{tP}, \qquad \text{with } t \mapsto U(t) \text{ is } T\text{-periodic}, \tag{2.3}$$
and P is in $\mathcal{M}_N(\mathbb{C})$.

Proof. Let us define $S(t) := R(t+T)R(T)^{-1}$. We check that $S(0) = \mathrm{Id}$ and that S satisfies the same equation as R, namely
$$S'(t) = R'(t+T)R(T)^{-1} = A(t+T)R(t+T)R(T)^{-1} = A(t)S(t).$$

By the uniqueness statement of Cauchy-Lipschitz, we see that $S(t) = R(t)$ for all t, hence (2.2) is true.

To find P, and U we observe that if (2.3) is true, we must have

$$U(0) = R(0) = \mathrm{Id}, \quad U(T) = U(0) = \mathrm{Id}, \quad R(T) = e^{TP}.$$

We now use the following lemma:

Lemma 2.2. *The exponential map for $N \times N$ complex matrices* $\exp : \mathcal{M}_N(\mathbb{C}) \to \mathcal{GL}_N(\mathbb{C})$ *is onto. In other words, for every invertible $N \times N$ complex matrix, there exists a pre-image by the exponential map.*

Since $R(T)$ is invertible, we may find a pre-image by exp, and dividing this pre-image by T gives us a choice of P. Once P is fixed, we define for any t

$$U(t) := R(t)e^{-tP},$$

and it remains to show that U is indeed T-periodic (exercise!). □

Remark 2.3. *You can look for a proof of Lemma 2.2 online, there are essentially two approaches: a purely "linear algebraic" one using the Dunford decomposition, and a "differential calculus" one using the inverse function theorem and the fact that $\mathcal{GL}_N(\mathbb{C})$ is arc-connected. Let us observe that if M is an invertible matrix which is also* ***diagonalizable****, then it is easy to find B such that $e^B = M$. Indeed, we write*

$$M = Q\mathrm{diag}(z_1, \dots, z_N)Q^{-1}$$

with a certain change of basis matrix Q. Since M is invertible, all the z_i's are nonzero and thus there exists a complex number w_i such that $e^{w_i} = z_i$. Then we let

$$B := Q\mathrm{diag}(w_1, \dots, w_N)Q^{-1}.$$

The properties of the exponential of matrices imply that

$$e^B = Qe^{\mathrm{diag}(w_1,\dots,w_N)}Q^{-1} = Q\mathrm{diag}(e^{w_1}, \dots, e_{w_N})Q^{-1} = M.$$

Remark 2.4. *Theorem 1 shows that solutions are, in general, not periodic. However, the relation (2.2) implies that it is enough to know the resolvent $R(t)$ for $t \in [0, T]$, thereafter we can deduce $R(t)$ for all t*

3 Qualitative study from the resolvent

3.1 Generalities

In this section, we show that the knowledge of $R(T)$ (or, equivalently, of P) provides some information on the qualitative behavior of the solutions to (2.1) as $t \to +\infty$.

Proposition 3.1. *All the solutions go to 0 as $t \to +\infty$ if and only if all the eigenvalues of $R(T)$ belong to $\{|z| < 1\}$.*

If there is an eigenvalue whose modulus is strictly greater than 1, then there exists a solution whose norm tend to $+\infty$ as $t \to +\infty$.

Proof. We use the following fact: Let X be an eigenvector of $R(T)$ associated to an eigenvalue λ. Then

$$R(T)X = \lambda X, \quad \forall k \geq 1, R(kT) = \lambda^k X.$$

Exercise: complete the proof (you may assume that $R(T)$ is diagonalizable, that should help), by decomposing any X in a basis of eigenvectors, and observing that between $R(kT)$ and $R((k+1)T)$ the system evolves "in a bounded way". □

3.2 A theorem by Liouville

In dimension 2, the knowledge of the determinant tells us something strong about the eigenvalues. We will often encounter cases where the determinant of the resolvent is 1, due to the following result:

Theorem 2 (Liouville)**.** *If* $\mathrm{trace}(A(t))$ *is always* 0 *in* (2.1)*, then the resolvent* $R(t)$ *always has determinant* 1*.*

Proof. Of course, since $R(0) = \mathrm{Id}$, the resolvent has determinant 1 for $t = 0$. We compute

$$\frac{d}{dt}\det(R(t)) = \mathrm{trace}(R'(t)R(t)^{-1}) = \mathrm{trace}(A(t)R(t)R(t)^{-1}) = \mathrm{trace}(A(t)) = 0,$$

which proves the result. We have used the linear algebra fact that

$$\det(M + \epsilon H) \approx \det(M) + \epsilon \mathrm{trace}(HM^{-1}) + O(\epsilon^2)$$

look up "differential of the determinant" for a proof (or, better, try to prove it yourself!). □

This fact is used often in incompressible fluid dynamics, and is some stated as: "the flow of a divergence-free vector field is volume-preserving".

3.3 The two-dimensional case

Assume that $N = 2$ and that the assumptions of Lemma 2 are satisfied. Then the eigenvalues are either a couple λ, λ^{-1} of real numbers, or two complex numbers of the form $e^{i\pm\theta}$. In the first case, we have $|\mathrm{trace}(R(T))| > 2$, in the second case we have $|\mathrm{trace}(R(T))| < 2$, and there are two limit cases when $R = \pm\mathrm{Id}$, for which the trace is ± 2. In view of Proposition 3.1, we see that

- If $|\mathrm{trace}(R(T))| < 2$, the system is "stable" in the sense that all solutions converge to 0 as t gets large.
- If $|\mathrm{trace}(R(T))| > 2$, the system is "unstable" in the sense that there is a solution diverging to $+\infty$ as t gets large.

4 Hill to Mathieu to the swing

4.1 Hill equation

Let α, β be two parameters, and let φ be a fixed continuous function. We assume that φ is T-periodic and we consider the Hill equation

$$(H_{\alpha,\beta}) \quad x'' + V_{\alpha,\beta}(t)x = 0, \tag{4.1}$$

where $V_{\alpha,\beta}(t) = \alpha + \beta\varphi(t)$.

For $\beta = 0$, we obtain explicit solutions (exercise: find them!) and in particular we find

$$\operatorname{trace}(R_{\alpha,0}(T)) = \begin{cases} 2\cos(\sqrt{\alpha}T) & \text{if } \alpha > 0 \\ 2 & \text{if } \alpha = 0 \\ 2\cosh(\sqrt{-\alpha}T) & \text{if } \alpha < 0 \end{cases}$$

Thus we see that if $\alpha > 0$ and $\sqrt{\alpha}$ is not of the form $\frac{k^2\pi^2}{T^2}$ for an integer k, then

$$|\operatorname{trace}(R_{\alpha,0}(T))| < 2$$

and the system is "stable" in the previous sense.

It remains to extend these considerations to the case $\beta \neq 0$...