# Linear ODE's with periodic coefficients

## 1 Examples

- $y' = \sin(t)y$, solutions $Ce^{-\cos t}$. Periodic, go to 0 as $t \to +\infty$.
- $y' = -2\sin^2(t)y$, solutions $Ce^{-t-\sin(2t)/2}$. Not periodic, go to to 0 as $t \to +\infty$.
- $y' = (1+\sin(t))y$, solutions $Ce^{t-\cos(t)}$. Not periodic, do not go to 0 as $t \to +\infty$.

## 2 Floquet's theorem

We consider the ODE

$$Y' = A(t)Y, \tag{2.1}$$

where $t \mapsto A(t)$ is a continuous, $T$-periodic map from $(-\infty, +\infty)$ to $\mathcal{M}_N(\mathbb{R})$ ($N \times N$ matrices with real coefficients).

For any $t$ in $(-\infty, +\infty)$, we introduce the **resolvent** $R(t)$, which is nothing but the flow of the ODE, i.e. for $X$ in $\mathbb{R}^N$, we define $R(t)X$ as the value at time $t$ of the solution to (2.1) which is equal to $X$ at $t = 0$.

**Lemma 2.1** (Properties of the resolvent)**.** *1. $R(0) = \mathrm{Id}$ and $R(t)$ is always invertible.*

*2. For any $t$, $R(t)$ is a linear map from $\mathbb{R}^N \to \mathbb{R}^N$.*

*3. For any $t$, we have $R'(t) = A(t)R(t)$.*

*Proof.* 1. By definition, and by the fact that $R(t)R(-t) = R(0)$.

2. This follows from the linearity of the ODE.

3. Exercise.

□

**Theorem 1** (Floquet)**.** *We have, for any $t \in (-\infty, +\infty)$*

$$R(t+T) = R(t)R(T), \tag{2.2}$$

*and $R(t)$ can be written as*

$$R(t) = U(t)e^{tP}, \qquad \text{with } t \mapsto U(t) \text{ is } T\text{-periodic}, \tag{2.3}$$

*and $P$ is in $\mathcal{M}_N(\mathbb{C})$.*

*Proof.* Let us define $S(t) := R(t+T)R(T)^{-1}$. We check that $S(0) = \mathrm{Id}$ and that $S$ satisfies the same equation as $R$, namely

$$S'(t) = R'(t+T)R(T)^{-1} = A(t+T)R(t+T)R(T)^{-1} = A(t)S(t).$$

By the uniqueness statement of Cauchy-Lipschitz, we see that $S(t) = R(t)$ for all $t$, hence (2.2) is true.

To find $P$, and $U$ we observe that if (2.3) is true, we must have

$$U(0) = R(0) = \mathrm{Id}, \quad U(T) = U(0) = \mathrm{Id}, \quad R(T) = e^{TP}.$$

We now use the following lemma:

**Lemma 2.2.** *The exponential map for $N \times N$ complex matrices* $\exp : \mathcal{M}_N(\mathbb{C}) \to \mathcal{GL}_N(\mathbb{C})$ *is onto. In other words, for every invertible $N \times N$ complex matrix, there exists a pre-image by the exponential map.*

Since $R(T)$ is invertible, we may find a pre-image by exp, and dividing this pre-image by $T$ gives us a choice of $P$. Once $P$ is fixed, we define for any $t$

$$U(t) := R(t)e^{-tP},$$

and it remains to show that $U$ is indeed $T$-periodic (exercise!). □

**Remark 2.3.** *You can look for a proof of Lemma 2.2 online, there are essentially two approaches: a purely "linear algebraic" one using the Dunford decomposition, and a "differential calculus" one using the inverse function theorem and the fact that $\mathcal{GL}_N(\mathbb{C})$ is arc-connected. Let us observe that if $M$ is an invertible matrix which is also **diagonalizable**, then it is easy to find $B$ such that $e^B = M$. Indeed, we write*

$$M = Q\mathrm{diag}(z_1, \ldots, z_N)Q^{-1}$$

*with a certain change of basis matrix $Q$. Since $M$ is invertible, all the $z_i$'s are nonzero and thus there exists a complex number $w_i$ such that $e^{w_i} = z_i$. Then we let*

$$B := Q\mathrm{diag}(w_1, \ldots, w_N)Q^{-1}.$$

*The properties of the exponential of matrices imply that*

$$e^B = Qe^{\mathrm{diag}(w_1,\ldots,w_N)}Q^{-1} = Q\mathrm{diag}(e^{w_1}, \ldots, e_{w_N})Q^{-1} = M.$$

**Remark 2.4.** *Theorem 1 shows that solutions are, in general, not periodic. However, the relation (2.2) implies that it is enough to know the resolvent $R(t)$ for $t \in [0, T]$, thereafter we can deduce $R(t)$ for all $t$*

# 3 Qualitative study from the resolvent

## 3.1 Generalities

In this section, we show that the knowledge of $R(T)$ (or, equivalently, of $P$) provides some information on the qualitative behavior of the solutions to (2.1) as $t \to +\infty$.

**Proposition 3.1.** *All the solutions go to 0 as $t \to +\infty$ if and only if all the eigenvalues of $R(T)$ belong to $\{|z| < 1\}$.*

*If there is an eigenvalue whose modulus is strictly greater than 1, then there exists a solution whose norm tend to $+\infty$ as $t \to +\infty$.*

*Proof.* We use the following fact: Let $X$ be an eigenvector of $R(T)$ associated to an eigenvalue $\lambda$. Then

$$R(T)X = \lambda X, \quad \forall k \geq 1, R(kT) = \lambda^k X.$$

Exercise: complete the proof (you may assume that $R(T)$ is diagonalizable, that should help), by decomposing any $X$ in a basis of eigenvectors, and observing that between $R(kT)$ and $R((k+1)T)$ the system evolves "in a bounded way". □

### 3.2 A theorem by Liouville

In dimension 2, the knowledge of the determinant tells us something strong about the eigenvalues. We will often encounter cases where the determinant of the resolvent is 1, due to the following result:

**Theorem 2** (Liouville)**.** *If* trace$(A(t))$ *is always* 0 *in* (2.1)*, then the resolvent* $R(t)$ *always has determinant* 1*.*

*Proof.* Of course, since $R(0) = \mathrm{Id}$, the resolvent has determinant 1 for $t = 0$. We compute

$$\frac{d}{dt}\det(R(t)) = \mathrm{trace}(R'(t)R(t)^{-1}) = \mathrm{trace}(A(t)R(t)R(t)^{-1}) = \mathrm{trace}(A(t)) = 0,$$

which proves the result. We have used the linear algebra fact that

$$\det(M + \epsilon H) \approx \det(M) + \epsilon \mathrm{trace}(HM^{-1}) + O(\epsilon^2)$$

look up "differential of the determinant" for a proof (or, better, try to prove it yourself!). □

This fact is used often in incompressible fluid dynamics, and is some stated as: "the flow of a divergence-free vector field is volume-preserving".

### 3.3 The two-dimensional case

Assume that $N = 2$ and that the assumptions of Lemma 2 are satisfied. Then the eigenvalues are either a couple $\lambda, \lambda^{-1}$ of real numbers, or two complex numbers of the form $e^{i\pm\theta}$. In the first case, we have $|\mathrm{trace}(R(T))| > 2$, in the second case we have $|\mathrm{trace}(R(T))| < 2$, and there are two limit cases when $R = \pm\mathrm{Id}$, for which the trace is $\pm 2$. In view of Proposition 3.1, we see that

- If $|\mathrm{trace}(R(T))| < 2$, the system is "stable" in the sense that all solutions converge to 0 as $t$ gets large.
- If $|\mathrm{trace}(R(T))| > 2$, the system is "unstable" in the sense that there is a solution diverging to $+\infty$ as $t$ gets large.

## 4 Hill to Mathieu to the swing

### 4.1 Hill equation

Let $\alpha, \beta$ be two parameters, and let $\varphi$ be a fixed continuous function. We assume that $\varphi$ is $T$-periodic and we consider the Hill equation

$$(H_{\alpha,\beta}) \quad x'' + V_{\alpha,\beta}(t)x = 0, \tag{4.1}$$

where $V_{\alpha,\beta}(t) = \alpha + \beta\varphi(t)$. We denote by $R_{\alpha,\beta}$ the resolvent with parameters $\alpha, \beta$.

For $\beta = 0$, we obtain explicit solutions (exercise: find them!) and in particular we find

$$\text{trace}\,(R_{\alpha,0}(T)) = \begin{cases} 2\cos(\sqrt{\alpha}T) & \text{if } \alpha > 0 \\ 2 & \text{if } \alpha = 0 \\ 2\cosh(\sqrt{-\alpha}T) & \text{if } \alpha < 0 \end{cases}$$

Thus we see that if $\alpha > 0$ and $\alpha$ is not of the form $\frac{k^2\pi^2}{T^2}$ for an integer $k$, then

$$|\text{trace}\,(R_{\alpha,0}(T))| < 2$$

and the system is "stable" in the previous sense. The boundary cases between "stable"/"unstable" are the $\alpha = \frac{k^2\pi^2}{T^2}$ for some $k$.

It remains to extend these considerations to the case $\beta \neq 0$. This relies on a version "with parameters" of Cauchy-Lipschitz.

## 4.2 Cauchy-Lipschitz and fixed point with parameters

**Theorem 3** (Cauchy-Lipschitz with parameters)**.** *Let us consider the family of ODE's*

$$X' = F(t, X, \gamma), \quad X(t_0) = X^0,$$

*where $\gamma$ is some parameter*[1] *We denote by $X_\gamma$ the solution to this ODE (it depends on $\gamma$ because the ODE depends on $\gamma$).*

*Roughly speaking, Cauchy-Lipschitz with parameters ensures that if $F$ is regular enough (typically, $C^p$) in $t, X$ and depends on $\gamma$ in a $C^p$ way, then $X_\gamma$ depends on $\gamma$ in a $C^p$ way.*

*You can read Section 2.5. of the "textbook", in particular Theorem 2.12.*

We will state and prove the key ingredient for Theorem 3 in the $C^0$ regularity: a fixed point theorem with parameters.

**Theorem 4** (Picard's fixed point theorem with parameters)**.** *Let $(X, d)$ be a metric space ($d$ is the distance on $X$) and let $\{f_\gamma\}_\gamma$ be a family of maps from $X$ to $X$ depending on a parameter $\gamma$, that satisfies, for some $k < 1$, the contractivity property:*

$$\forall\gamma, \forall(x, y) \in X \times X, \quad d(f_\gamma(x), f_\gamma(y)) \leq kd(x, y).$$

*Hence for all $\gamma$, the map $f_\gamma$ is contractive on $X$, and thus by Picard's fixed point theorem it admits a unique fixed point. We denote by $x_\gamma$ this fixed point, such that*

$$f_\gamma(x_\gamma) = x_\gamma.$$

*We claim that if the family of functions $\{f_\gamma\}_\gamma$ depends continuously on $\gamma$, then $x_\gamma$ depends continuously on $\gamma$.*

[1] It can be just one real number, or a vector of parameters as in the example above where we have a family of ODE's depending on $\alpha, \beta$.

Remark: to precise what we mean by "$f_\gamma$ depends continuously on $\gamma$", we would need a notion of topology on the space of functions $X \to X$. It is perhaps easier to consider the problem as the data of a function

$$f : (\Gamma \times X) \to X$$

where $\Gamma$ is the space of parameters, and for any $\gamma$ in $\Gamma$ we let

$$f_\gamma := f(\gamma, \cdot) : X \to X.$$

In other words, we assume that $f_\gamma$ is always $k$-contractive for a fixed $k$ independent on $\gamma$, and we assume that the fucntion $f$ depends continuously on the first variable (the one in $\Gamma$) in the usual sense.

*Proof.* Let $\gamma$ in $\Gamma$ be fixed. We want to prove that

$$\lim_{\lambda \to \gamma} x_\lambda = x_\gamma,$$

which would prove that the fixed point depends continuously on the parameter. More precisely, we want to prove that

$$d(x_\lambda, x_\gamma) \to 0 \text{ as } \lambda \to \gamma.$$

For that, we first use the fact that $x_\lambda$ (resp. $x_\gamma$) is a fixed point of $f_\lambda$ (resp. $f_\gamma$) and write

$$d(x_\lambda, x_\gamma) = d(f_\lambda(x_\lambda), f_\gamma(x_\gamma)).$$

Next, we use the triangular inequality and write

$$d(f_\lambda(x_\lambda), f_\gamma(x_\gamma)) \leq d(f_\lambda(x_\lambda), f_\lambda(x_\gamma)) + d(f_\lambda(x_\gamma), f_\gamma(x_\gamma)).$$

The first term in the right-hand side can be bounded using the contractivity of $f_\lambda$, we have

$$d(f_\lambda(x_\lambda), f_\lambda(x_\gamma)) \leq k d(x_\lambda, x_\gamma),$$

and thus we obtain

$$d(x_\lambda, x_\gamma) \leq k d(x_\lambda, x_\gamma) + d(f_\lambda(x_\gamma), f_\gamma(x_\gamma)),$$

which can be re-written (since $k < 1$) as

$$d(x_\lambda, x_\gamma) \leq \frac{1}{1-k} d(f_\lambda(x_\gamma), f_\gamma(x_\gamma)).$$

The right-hand side can be expressed as

$$d\left(f(\lambda, x_\gamma), f(\gamma, x_\gamma)\right),$$

which tends to 0 as $\lambda \to \gamma$ because $f$ is continuous in the first variable. This concludes the proof. $\square$

### 4.3 Perturbations considerations

From Cauchy-Lipschitz with parameters, we easily deduce

**Lemma 4.1.** *The quantity* $(\alpha, \beta) \mapsto \operatorname{trace}(R_{\alpha,\beta}(T))$ *is continuous.*

In particular, if $(\alpha, 0)$ is inside a stability (resp. unstability) region, characterized by $\operatorname{trace}(R_{\alpha,0}(T)) < 2$ (resp. $> 2$) then for $\beta$ small enough the inequality $\operatorname{trace}(R_{\alpha,\beta}(T)) < 2$ (resp. $< 2$) will remain true, and thus the system remains stable (resp. unstable). We are thus interested to understand what happens near the boundary between stable and unstable regions, and we will see that the presence of a $\beta \neq 0$ may change the expected behavior.

**Perturbative expansion** For this, we need to have a more precise understanding on the dependency of $\operatorname{trace}(R_{\alpha,\beta}(T))$ with respect to the parameters. The result of Cauchy-Lipschitz with parameters guarantees that we may find an asymptotic expansion of this quantity as a power series in $\beta$. How do we find it?

Let us observe that $R(T)$ can be computed by finding the value at time $T$ of any two independent solutions of the ODE (because they form a basis), and it is usually convenient to look at the solutions $U_{\alpha,\beta}$, $V_{\alpha,\beta}$ defined by

$$\begin{cases} U_{\alpha,\beta}(0) &= 1 \\ U'_{\alpha,\beta}(0) &= 0 \end{cases} \quad \begin{cases} V_{\alpha,\beta}(0) &= 0 \\ V'_{\alpha,\beta}(0) &= 1 \end{cases}$$

We look for expansions of $U_{\alpha,\beta}, V_{\alpha,\beta}$ in powers of $\beta$, as

$$U_{\alpha,\beta}(t) = U_{\alpha,0}(t) + \beta R_{1,\alpha}(t) + \beta^2 R_{2,\alpha}(t) + \cdots$$

The zeroth-order term can be computed by solving the ODE with $\beta = 0$, we obtain for example

$$U_{\alpha,0}(t) = \cos(\sqrt{\alpha}t).$$

Then the next-order coefficients are obtained recursively by solving

$$R''_{N,\alpha} + \alpha R_{N,\alpha} = -\varphi(t) R_{N-1,\alpha},$$

where $\varphi$ is the function appearing in the initial ODE. For some particular forms of $\varphi$, this is doable - at least for a computer.

### 4.4 Mathieu equation

In the case $\varphi(t) = \cos(2t)$, the period is $T = \pi$, and the expansion above can be computed. We obtain e.g.

- Order $O(\beta^4)$

$$\operatorname{trace}(R_{\alpha,\beta}(T)) = 2\cos(\pi\sqrt{\alpha}) + \beta^2 \frac{\pi}{8} \frac{\sin(\pi\sqrt{\alpha})}{(\alpha-1)\sqrt{\alpha}} + O(\beta^4)$$

- Order $O(\beta^6)$

$$\begin{aligned} \operatorname{trace}(R_{\alpha,\beta}(T)) = 2\cos(\pi\sqrt{\alpha}) + \beta^2 \frac{\pi}{8} \frac{\sin(\pi\sqrt{\alpha})}{(\alpha-1)\sqrt{\alpha}} \\ + \beta^4 \left[ \frac{-\pi^2}{256\alpha(\alpha-1)^2} \cos(\pi\sqrt{\alpha}) + \frac{\pi}{512} \frac{(15\alpha^2 - 35\alpha + 8)\sin(\pi\sqrt{\alpha})}{(\alpha-4)(\alpha-1)^3\alpha^{3/2}} \right] + O(\beta^6) \end{aligned}$$

- Close to $\alpha \approx 0$ and for $\beta$ small, we have

$$\operatorname{trace}\left(R_{\alpha,\beta}(T)\right) \approx 2 - \pi^2\alpha - \frac{\pi^2}{8}\beta^2.$$

  If $\alpha$ is small but negative, we would expect the system to be unstable, but if $\beta$ satisfies

$$\beta^2 > 8|\alpha|,$$

  then the trace is $< 2$ and thus we have stability.
- Close to $\alpha \approx 1$, and for $\beta$ small, we have

$$\operatorname{trace}\left(R_{\alpha,\beta}(T)\right) \approx -2 + \frac{\pi^2}{4}(\alpha-1)^2 - \frac{\pi^2}{16}\beta^2.$$

  If $\beta > 2(\alpha - 1)$, the trace is $> 2$ and thus we have unstability, even though we are in the zone $\alpha > 0$.

### 4.5 Swing, inverted pendulum

This can be used as a toy model to explain a swing (stretching one's legs introduced a periodic variation of the length of the swing, and for a good choice of the parameters the equilibrium position becomes unstable) or an inverted pendulum (Kapitsa pendulum: the unstable equilibrium becomes stable).