Fixed point theorem and Cauchy-Lipschitz for linear ODE's

February 7, 2018

1 Fixed point theorem in complete metric spaces

Definition 1.1 (Metric space). *Let X be a set and d a function from X to $[0, +\infty)$. We say that d is a distance/metric on X when*

- *For all x, y in X, $d(x, y) = d(y, x)$.*
- *For all x, y, z in X, $d(x, y) \leq d(x, z) + d(z, y)$ (triangle inequality).*
- *We have $d(x, y) = 0$ if and only if $x = y$.*

If d is a distance on X, we say that (X, d) is a metric space.

Examples of metric spaces:

- $\mathbb{R}$ with the distance $d(x, y) = |x - y|$.
- $\mathbb{Q}$ with the distance $d(x, y) = |x - y|$.
- $\mathbb{R}^N$ with the distance $d(x, y) = \|x - y\|$.
- The Earth with the geodesic ("as the crow flies") distance.
- $\mathbb{Z}^2$ with the "Manhattan distance" (see "Taxicab geometry" on Wikipedia).
- The space $C^0([0, 1])$ of real-valued continuous functions on $[0, 1]$, with the distance

$$d(f, g) := \sup_{x \in [0,1]} |f(x) - g(x)|,$$

  which is usually denoted by $\|f - g\|_\infty$ (the "sup norm" or "uniform norm").

Definition 1.2 (Lipschitz functions). *Let (X, d_X) and (Y, d_Y) be two metric spaces and $F : X \to Y$. We say that F is k-Lipschitz when we have for all a, b in X*

$$d_Y(F(a), F(b)) \leq k d_X(a, b).$$

If $k < 1$, we say that F is a contraction.

Definition 1.3 (Cauchy sequence). *Let (X, d) be a metric space and $x = \{x_n\}_n$ be a sequence of points in X. We say that x is a Cauchy sequence when*

$$\forall \epsilon > 0, \exists M \geq 0, \forall m, n \geq M, d(x_n, x_m) \leq \epsilon.$$

Definition 1.4 (Complete space). *We say that (X, d) is a complete metric space when every Cauchy sequence is convergent.*

Examples of complete spaces: all the examples above, except $\mathbb{Q}$ which is not a complete metric space when endowed with the usual distance.

Proposition 1.5 (Picard's fixed point theorem)**.** *Let (X, d) be a complete metric space and $F : X \to X$ be a contraction, then there exists a unique fixed point for F in X, i.e. there exists a unique point x in X such that $F(x) = x$.*

Proof. Since F is a contraction, we can find $k < 1$ such that for any a, b in X,

$$d(F(a), F(b)) \leq kd(a, b).$$

Uniqueness is easy: if x, y are two fixed points, we have

$$d(x, y) = d(F(x), F(y)) \leq kd(x, y),$$

with $k < 1$, which is impossible except if $d(x, y) = 0$, which implies that $x = y$, so there is at most one fixed point.

Existence. Let us define a sequence as follows: pick any point x_0 in X, and define x_n by induction: for $n \geq 0$, we let $x_{n+1} := F(x_n)$. By assumption on F we have for any $n \geq 1$

$$d(x_n, x_{n+1}) = d(F(x_{n-1}), F(x_n)) \leq kd(x_{n-1}, x_n).$$

It is then easy to check that for $n \geq 0$,

$$d(x_n, x_{n+1}) \leq k^n d(x_0, x_1), \tag{1.1}$$

in other words, the successive distances are shrinking exponentially fast. It implies that $\{x_n\}_n$ is a Cauchy sequence (exercise!) and since X is complete, we deduce that $\{x_n\}_n$ converges to some point x in X. Passing to the limit $n \to \infty$ (exercise: why can we?) in the equation

$$x_{n+1} = F(x_n)$$

we see that $x = F(x)$ and thus x is a fixed point for F. □

Remark: if X is a vector space, we can write

$$x_{n+1} = x_0 + (x_1 - x_0) + (x_2 - x_1) + \cdots + (x_{n+1} - x_n),$$

and the convergence of $\{x_n\}_n$ can be interpreted as the convergence of the series

$$\sum_{k=0}^{+\infty} x_{k+1} - x_k.$$

The estimate (1.1) shows that this series is *absolutely convergent*. A consequence of completeness is that in a complete normed vector space, any absolutely convergent series is convergent.

Proposition 1.6 (Clever Picard)**.** *Let (X, d) be a complete metric space and $F : X \to X$ be a map such that for some $l \geq 1$, the map $F^{\circ l}$ (the l-th iterate of F) is a contraction. Then there exists a unique fixed point for F on X.*

Proof. Applying Picard's theorem fo $F^{\circ l}$, we find a unique fixed point for $F^{\circ l}$. Let x be this fixed point, we have $F^l(x) = x$. In particular, we have $F(x) = F(F^l(x)) = F^l(F(x))$ hence $F(x)$ is a fixed point of F^l. Since we know that there is a *unique* fixed point for F^l, namely x, we must have $F(x) = x$, hence x is a fixed point for F. This proves existence of a fixed point for F. Uniqueness is obtained by a similar trick. □

2 Application to linear ODE's

Our goal here is to prove the Cauchy-Lipschitz theorem in the linear case.

Theorem 1 (Cauchy-Lipschitz, linear case)**.** *Let I be an open interval of $\mathbb{R}$, let $N \geq 1$ and let $t \mapsto A(t)$ be a continuous function from I to $M_{N\times N}(\mathbb{R})$ and $t \mapsto B(t)$ be a continuous function from t to $\mathbb{R}^N$. Let t_0 be in I and let Y_0 be in $\mathbb{R}^N$. There exists a unique solution defined on I to the linear ODE with initial condition*

$$Y' = A(t)Y + B(t), \quad Y(t_0) = Y_0. \tag{2.1}$$

Proof. We want to apply the fixed point theorem, and thus to re-write the ODE as a fixed point problem. Let $[a, b]$ be a line segment included in I, containing t_0. By integrating (2.1) we see that Y is a solution on (a, b) if and only if, for any t in (a, b), we have

$$Y(t) - Y(t_0) = \int_{t_0}^{t} A(s)Y(s)ds + \int_{t_0}^{t} B(s)ds.$$

Let X be the space of continuous functions on $[a, b]$ with values in $\mathbb{R}^N$ and which have the value Y_0 at t_0. We turn X into a metric space by using the "sup norm" as above

$$d(Y, \tilde{Y}) := \sup_{t\in[a,b]} \|Y(t) - \tilde{Y}(t)\|.$$

It is a classical fact that we obtain a *complete* metric space (you can try to think of a proof of the completeness). We define F on X by setting, for t in $[a, b]$

$$F(Y)(t) := Y_0 + \int_{t_0}^{t} A(s)Y(s)ds + \int_{t_0}^{t} B(s)ds.$$

It defines a map from X to X (exercise: why?) and if Y is a fixed point for F then Y is a solution of (2.1) on (a, b). In order to apply Proposition 1.5 or Proposition 1.6, we need to see if F (or one of its iterates) is a contraction. Let us chose $Y, \tilde{Y}$ in X and compute the distance $d(F(Y), F(\tilde{Y}))$. We have

$$d(F(Y), F(\tilde{Y})) = \sup_{t\in[a,b]} \|F(Y)(t) - F(\tilde{Y})(t)\|,$$

so we compute, for any t in $[a, b]$

$$F(Y)(t) - F(\tilde{Y})(t) = \int_{t_0}^{t} A(s)\left(Y(s) - \tilde{Y}(s)\right) ds.$$

Since $s \mapsto A(s)$ is a continuous, matrix-valued map, there exists a constant C such that for any s in $[a, b]$ and any vector U in $\mathbb{R}^N$ we have

$$\|A(s)U\| \leq C\|U\|. \tag{2.2}$$

We may thus write

$$\|F(Y)(t) - F(\tilde{Y})(t)\| \leq (t - t_0)C \sup_{s\in[a,b]} \|Y(s) - \tilde{Y}(s)\|,$$

which implies that

$$\sup_{t\in[a,b]} \|F(Y)(t) - F(\tilde{Y})(t)\| \leq C(b-a) \sup_{s\in[a,b]} \|Y(s) - \tilde{Y}(s)\|,$$

and thus $d(F(Y), F(\tilde{Y})) \leq C(b-a)d(Y, \tilde{Y})$.

If it happens that $C(b-a) < 1$, then F is a contraction and we are done. In general, however, we need to study the iterates of F. For example, we can write

$$F^{\circ 2}(Y)(t) - F^{\circ 2}(\tilde{Y})(t) = \int_{t_0}^{t} A(s) \left(\int_{t_0}^{s} A(u) \left(Y(u) - \tilde{Y}(u) \right) du \right) ds$$

(exercise: check that it is correct!). We obtain, by using (2.2) twice

$$\|F^{\circ 2}(Y)(t) - F^{\circ 2}(\tilde{Y})(t)\| \leq C^2 \sup_{u\in[a,b]} \|Y(u) - \tilde{Y}(u)\| \int_{t_0}^{t} |s - t_0| ds$$

(exercise: check that it is correct). We thus get

$$\sup_{t\in[a,b]} \|F^{\circ 2}(Y)(t) - F^{\circ 2}(\tilde{Y})(t)\| \leq \frac{C^2(b-a)^2}{2} \sup_{u\in[a,b]} \|Y(u) - \tilde{Y}(u)\|,$$

which means that

$$d(F^{\circ 2}(Y), F^{\circ 2}(\tilde{Y})) \leq \frac{C^2(b-a)^2}{2} d(Y, \tilde{Y}).$$

By induction, we would show similarly, for any $l \geq 1$

$$d(F^{\circ l}(Y), F^{\circ l}\tilde{Y})) \leq \frac{C^l(b-a)^l}{l!} d(Y, \tilde{Y}).$$

Since the quantity $\frac{C^l(b-a)^l}{l!}$ goes to zero as l goes to infinity, it must be strictly less than one for l large enough. This ensures that one of the iterates of F is a contraction, and Proposition 1.6 implies that there exists a fixed point for F, hence a solution to (2.1) on (a,b). Uniqueness of the solution can either be deduced from the uniqueness of the fixed point (with a bit of carefulness) or by a simple application of Grönwall's lemma.

Since this is true for every $(a,b) \subset I$, we may find a unique solution defined on the whole interval I. (Exercise: make this conclusion rigorous. In particular, if Y is a solution on an interval J and $\tilde{Y}$ is a solution on $K \subset J$, why do Y and $\tilde{Y}$ necessarily coincide on K?) □