Maximal solutions and the Cauchy-Lipschitz's theorem

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1 Maximal solutions

Let us consider a general ODE, with the unknown function Y valued in $\mathbb{R}^N$, of the form

$$Y' = F(t, Y), \tag{1.1}$$

where F is defined on $I \times \Omega$, with I an interval of $\mathbb{R}$ and Ω an (open) subset of $\mathbb{R}^N$.

We recall the definition of solutions.

Definition 1.1 (Solution)**.** *A solution to (1.1) is the data of an (open) interval J included in I and of a function $Y : J \to \mathbb{R}^N$ which is differentiable on J and such that*

$$Y'(t) = F(t, Y(t)) \quad \text{for all } t \text{ in } J.$$

Implicitly, we also require that $Y(t)$ belongs to Ω for all t in J, otherwise $F(t, Y(t))$ does not make sense.

Let us emphasize that a solution is always the data of a function **plus** an interval where we consider it as satisfying (1.1). In particular,

1. $t \mapsto e^t$ on $[3, 189]$
2. $t \mapsto e^t$ on $[-1, 2018]$
3. $t \mapsto \pi e^t$ on $(-\infty, +\infty)$

are three **different** solutions to the ODE $y' = y$. Of course, the third one is "more different" from the others than 1. and 2. are different from each other. In fact, the second one extends the first one, in the following sense.

Definition 1.2 (Extension of a solution)**.** *Let Y_1, defined on I_1 and Y_2, defined on I_2 be two solutions to the same ODE (1.1). We say that Y_2 is an extension of Y_1 if I_2 contains I_1 and if the function Y_1 is equal to Y_2 on I_1.*

Definition 1.3 (Maximal solution)**.** *A solution is said to be* ***maximal*** *when it cannot be extended (in the sense of the previous definition).*

The Cauchy-Lipschitz's theorem states that, under the correct assumptions of regularity of F (that is: F is continuous with respect to t and locally Lipschitz with respect to Y), there is existence and uniqueness of a maximal solution with given initial condition.

What it means is that, for any t_0 in I and Y_0 in Ω, fixed, there exists a unique solution Y to (1.1), defined on some interval $J \subset I$, which is such that $Y(t_0) = Y_0$ and which is maximal in the previous sense.

Let K be a sub-interval of J containing t_0. Let us denote by $\tilde{Y}_K$ the restriction of Y to K. Then $\tilde{Y}_K$, defined on K, is **another solution** to (1.1) with the correct initial condition, but it is **not maximal** because it can be extended by Y on J. The Cauchy-Lipschitz's theorem ensures that this is the only possibility: *any solution to* (1.1) *taking the value* Y_0 *at time* t_0 *can be extended by* Y *on* K*, in other words they are all restrictions of the maximal solution to some sub-interval.*

2 A counter-example if F is not locally Lipschitz

Let us consider the ODE $y' = 3(y^2)^{1/3}$. Here $F(t, y) = 3(y^2)^{1/3}$, with $I = \mathbb{R}$ (in fact, the right-hand side does not depend on t) and $\Omega = (-\infty, +\infty)$. Let us make the following observation right away: F is not locally Lipschitz on I. Indeed, the function $x \mapsto (y^2)^{1/3}$ is C^1 and thus locally Lipschitz on $(-\infty, 0)$, on $(0, +\infty)$, but not at 0. We will see that this can lead to non-uniqueness of (maximal) solutions.

Exercise: for any $K \leq 0$, we introduce the function y_K defined piece-wise by

$$y_K(t) = \begin{cases} (t-K)^3 & \text{if } t \leq K, \\ 0 & \text{if } K < t < 0 \\ t^3 & \text{if } t \geq 0 \end{cases}$$

Check that they are all solutions to the ODE, they are all defined on $\mathbb{R}$ (and are thus maximal) and they all satisfy the same initial condition $y_K(0) = 0$.

3 How to check maximality

Let us assume that we have found some solution Y defined on J to the ODE (1.1).

- Is J equal to I? If yes, there is no way we can extend Y further, and thus the solution is maximal.
- Let a be an endpoint of J. Does $Y(t)$ has a limit when $t \to a$? If no, there is no way we can extend Y further than a, and thus the solution is maximal (in that direction, at least).
- If $Y(t)$ has a limit as $t \to a$, we have to be more careful. When solving the ODE, did we make any extra assumption (e.g. "the solution is positive" or "the solution does not vanish"?). If not, then we have found the maximal solution. If we did, we have to check whether there is no way to extend our solution further than a by removing this extra assumption (maybe Y changes its sign at a and is extended by a negative function, etc.)