HW 5 - some solutions

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Time of existence

1. We consider the ODE
$$y' = 36\cos\left(\sqrt{1+y^2}\right),$$
where y is an unknown real-valued function. Show that the maximal solutions are global, i.e. defined on $\mathbb{R}$.

   Let us introduce two functions $f(x) := 36\cos\left(\sqrt{1+x^2}\right)$ and $g(x) = 36$. For any real x we have $f(x) \leq g(|x|)$. The ODE that we consider can be written as $y' = f(y)$, and by the consequence of Grönwall's lemma mentioned in class, the solution of $y' = f(y)$ (say, with initial condition $y(0) = y_0$) is bounded by the solution of $y' = g(y)$ with the same initial condition. Since the solutions of $y' = 36$ are affine maps, they exist for all times, and thus so are the solutions of the ODE $y' = 36\cos\left(\sqrt{1+y^2}\right)$.

2. Same question for the ODE
$$y' = 36\sqrt{1+y^2}.$$

   Here we introduce $f(x) = 36\sqrt{1+x^2}$ and we observe that, letting
$$g(x) := 36\left(1+|x|\right),$$
   we have $|f(x)| \leq g(|x|)$ for any real x (because $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$ for $a, b \geq 0$). The solutions to $y' = g(x)$ are defined for all times, hence so are the solutions to $y' = f(x)$.

3. Now we consider the ODE
$$y' = 36(1+y^2)^{3/5},$$
with $y(0) = 1$. Give a lower bound on the time of existence of the maximal solution.

   We set $f(x) = 36(1+x^2)^{3/5}$ and we let $g(x) = 36(1+|x|^{6/5})$. Let us prove that $f(x) \leq g(x)$ for all x. It is enough to show that
$$(1+a)^{3/5} \leq (1+a^{3/5})$$
   for any real $a \geq 0$, which can be obtained e.g. by computing the derivative of both functions (and observing that they are equal for $a = 0$). For any y_0 fixed, the solution to the ODE
$$y' = f(y), \quad y(0) = y_0$$

is thus bounded by the solution to the ODE

$$y' = g(y) = 36(1 + |y|^{6/5}), \quad y(0) = y_0.$$

First, we solve the associated homogeneous equation $z' = 36|z|^{6/5}$. There is a constant solution equal to 0. If $z(0) > 0$, we can easily solve and find

$$-5\left(\frac{1}{z(t)^{1/5}} - \frac{1}{z(0)^{1/5}}\right) = 36t,$$

hence

$$z(t) = \left(\frac{1}{z(0)^{1/5}} - \frac{36}{5}t\right)^{-5}$$

which is defined on $(-\infty, T_c)$ with a blow-up time

$$T_c = \frac{5}{36z(0)^{1/5}}.$$

We can then check that this is the same time of existence for the solution to the non-homogeneous equation. This provides a lower bound on the time of existence of the solution to the original ODE.

Conserved quantities We consider the following ODE (with x the unknown function)

$$x'' + x + x^3 = 0, \tag{1}$$

which is an autonomous, second-order scalar ODE.

1. Find a conserved quantity, i.e. find a function Q on $\mathbb{R} \times \mathbb{R}$ such that, if x is a solution to (1) defined on I, we have

   $$Q(x(t), x'(t)) = \text{constant for } t \text{ in } I.$$

   Hint: for these questions, it is often fruitful to multiply the ODE by x' and to integrate.

   Following the trick, we get

   $$x''x' + xx' + x^3x' = 0$$

   hence the quantity

   $$Q(x, x') = \frac{1}{2}(x')^2 + \frac{1}{2}x^2 + \frac{1}{4}x^4$$

   is conserved.

2. Show that the maximal solutions to (1) are global, i.e. they are all defined on $\mathbb{R}$.

   Since $Q(x, x')$ is conserved, the maximal solutions live on a level set of Q. It is easy to check that these level sets are bounded, hence there is no blow-up in finite time of (x, x'), which implies that the solutions are defined for all time.

3. We want to prove that every solution is periodic. Let x_0, x_0' be in $\mathbb{R}$ and let x be the solution to (1) defined on $\mathbb{R}$ and satisfying $x(0) = x_0$ and $x'(0) = x_0'$.

(a) Prove that if there exists $T > 0$ such that $x(T) = x_0$ and $x'(T) = x'_0$, then for all t in $\mathbb{R}$ we have $x(t+T) = x(t)$ and $x'(t+T) = x'(T)$, and thus the solution is periodic.

This follows from the uniqueness result of Cauchy-Lipschitz. More precisely $t \mapsto x(t)$ and $t \mapsto x(t+T)$ are both solutions of the ODE with the same initial conditions hence are equal.

(b) Prove that there exists $T > 0$ such that $x(T) = x_0$ and $x'(T) = x'_0$. *Hint: you may need to use the result of question 1. Keep also in mind that x is continuous and real-valued.*

Since Q has been proven to be a conserved quantity in question 1., the orbit of $X_0 = (x_0, x'_0)$ under the flow $\{\Phi^t\}_t$ of our ODE is contained in the set

$$\mathcal{C} := \{(a,b), \quad Q(a,b) = Q(x_0, x'_0)\}.$$

Let us observe that either $X_0 = (0,0)$, in which case the solution is constant (hence periodic) or $(0,0)$ does not belong to $\mathcal{C}$. The set $\mathcal{C}$ is a closed, simple curve (that looks like an ellipsis) - it is an example of a *smooth planar algebraic curve*, and in this case it is not difficult to find a parametrization for $\mathcal{C}$. Let Γ be a continuous map from $[0,1]$ to $\mathcal{C}$ such that $\Gamma(0) = \Gamma(1) = X_0$ and Γ is one-to-one on $[0,1)$. We want to prove that

$$T = \inf\{t \in (0,+\infty), \quad \Phi^t(X_0) = X_0\} < +\infty.$$

Assume, for the sake of contradiction, that T is infinite, hence for all $t > 0$, $\Phi^t(X_0)$ belongs to $\mathcal{C} \setminus \{X_0\}$. For any $t > 0$, let C_t be defined as

$$C_t := \Gamma^{-1} \circ \Phi^t(X_0).$$

The map $t \mapsto C_t$ is a continuous map from $(0,+\infty)$ to $[0,1]$. We claim that it is one-to-one. Indeed, if $C_{t_1} = C_{t_2}$ with $0 < t_1 < t_2$ it means that $\Phi^{t_1}(X_0) = \Phi^{t_2}(X_0)$ (because by assumption neither of these points is X_0 and Γ^{-1} is one-to-one on $\mathcal{C} \setminus \{X_0\}$), hence $\Phi^{t_2 - t_1}(X_0) = X_0$, which yields a contradiction. Any continuous, one-to-one map from $(0,+\infty)$ to $[0,1]$ is monotonic and bounded, and thus admits a limit C_∞ in $[0,1]$ as $t \to \infty$. Hence $\Gamma(C_\infty)$ is a limit point of the flow, and thus must be a stationary point (this was proven in class) and must belong to $\mathcal{C}$, but the only stationary point for our ODE is $(0,0)$ and we ruled out the case where $(0,0)$ belongs to $\mathcal{C}$. Contradiction.

4. We now consider the equation

$$x'' + cx'x^2 + x^3 = 0,$$

where c is some constant. Are there periodic solutions?

Applying the same trick as above, we find that

$$\frac{d}{dt} Q(x(t), x'(t)) = -c(x')^2 x^2,$$

and hence Q is increasing/decreasing with t, depending on the sign of t. The constant solution $x \equiv 0$ is a periodic solution, but otherwise Q is strictly increasing/decreasing along an orbit, and thus there is no periodic solution.