

Stability

1 Definitions

We consider a general ODE $X' = F(t, X)$, with X an unknown function in \mathbb{R}^N . We fix a time t_0 and we let $\Phi_{t_0}^t$ be the flow at time t starting at t_0 . In other words, if x is in \mathbb{R}^N , $\Phi_{t_0}^t(x)$ is the value at time t of the solution to the ODE with initial condition $X(t_0) = x$. For simplicity, we will assume that all maximal solutions are global, or at least that they exist for all $t \geq t_0$, even though strictly speaking that is something that we could add to the definitions.

Fix x_0 in \mathbb{R}^N .

Stability We say that (t_0, x_0) is a stable initial condition, or that the associated solution is stable, if

$$\forall \epsilon > 0, \quad \exists \eta > 0, \quad \forall x \in \mathbb{R}^N, \quad \|x - x_0\| \leq \eta \implies \left(\forall t \geq t_0, \|\Phi_{t_0}^t(x) - \Phi_{t_0}^t(x_0)\| \leq \epsilon \right) \quad (1.1)$$

In other words, if x is chosen close enough to x_0 , we can guarantee that at all times $t \geq t_0$, the solution starting from x will stay arbitrarily close to the solution starting from x_0 .

Asymptotic stability We say that (t_0, x_0) is an asymptotically stable initial condition if **it is stable** and

$$\exists \delta > 0, \quad \forall x \in \mathbb{R}^N, \quad \|x - x_0\| \leq \delta \implies \lim_{t \rightarrow +\infty} \|\Phi_{t_0}^t(x) - \Phi_{t_0}^t(x_0)\| = 0. \quad (1.2)$$

The condition (1.2) can be interpreted by saying that any solution starting close enough to x_0 will eventually “converge back” to the solution starting at x_0

Some comments about the definitions:

- The δ in (1.2) and the η in (1.1) can be two different quantities.
- Think of small ball rolling in a parabola-shaped well (or a pendulum, or...). There is a solution given by “staying at the bottom of the well for all times”. Is this “equilibrium solution” stable? If you start close from the bottom, the ball will oscillate and will never get higher than where it started from, so (1.1) is satisfied - the equilibrium is stable. If there is no friction, the oscillations will have a constant amplitude, the ball does not slow down, so (1.2) is not satisfied and the equilibrium is not asymptotically stable. In the presence of friction, however, the oscillations get smaller and the ball will converge to the bottom of the well, so (1.1) is satisfied and the equilibrium is asymptotically stable.
- It might not be obvious to see why (1.2) does **not** imply (1.1). It is a good exercise to invent a situation where (1.2) holds but not (1.1).

2 Liapounov function

Let $X' = F(X)$ be an autonomous ODE, X_0 be a stationary point ($F(X_0) = 0$). Let Φ^t be the flow of this ODE at time t (starting from 0 - since the system is autonomous, the choice of the initial time is irrelevant).

A Liapounov function for the equilibrium X_0 is a function L defined on some compact neighborhood U of X_0 and with real values, satisfying:

1. The unique minimizer of L is X_0 .
2. For any $X \neq X_0$ in U , the quantity $L(\Phi^t(X))$ is decreasing in t .

Theorem 1. *The existence of a Liapounov function implies asymptotic stability of the equilibrium X_0 .*

Proof. Without loss of generality, we can suppose and $L(X_0) = 0$. For any c , we denote by \mathcal{L}_c the set

$$\mathcal{L}_c := \{x \in U, L(x) < c\}.$$

We also let $B(x, r)$ be the ball (interval if $N = 1$, disk if $N = 2$, ball for N general) of radius r around x .

We claim the following:

$$\forall \epsilon > 0, \exists c > 0, \mathcal{L}_c \subset B(X_0, \epsilon). \tag{2.1}$$

By contradiction, if it were not true, we would get a sequence of points outside a certain ball around X_0 at which L would take arbitrarily small values, hence by continuity we would get a zero of L distinct from X_0 which is ruled out by assumption (X_0 is the unique minimizer of L).

We can now prove stability. Let $\epsilon > 0$ and let c be such that (2.1) is satisfied. Since L is continuous, \mathcal{L}_c is open and contains X_0 , hence it contains a ball $B(X_0, \eta)$ for some $\eta > 0$. Now, if we start at a point X such that $\|X - X_0\| \leq \eta$, we are in $B(X_0, \eta)$, hence in \mathcal{L}_c , but since $L(\Phi^t(X))$ is decreasing in t we stay in \mathcal{L}_c for all times $t \geq 0$ and hence $\Phi^t(X)$ is in $B(X_0, \epsilon)$ for all t , so (1.1) is satisfied.

We now turn to prove asymptotic stability. Since stability is verified, we need to check (1.2). Pick X in U different from X_0 , close enough so that the sub-level set of L associated to $L(X)$ is compact. Since $L(\Phi^t(X))$ is decreasing in t and bounded below, this quantity converges to a real number.

- If $\lim_{t \rightarrow +\infty} L(\Phi^t(X)) = 0$, it is easy to prove that $\Phi^t(X)$ tends to X_0 because X_0 is the unique minimizer of L and L is continuous.
- Let us then assume that $\lim_{t \rightarrow +\infty} L(\Phi^t(X)) = l \neq 0$. We have $l < L(X)$ and since the sub-level set associated to $L(X)$ was chosen to be compact, we may find a sequence of times $t_n \rightarrow +\infty$ and a point X_1 such that
 1. $\lim_{n \rightarrow +\infty} \Phi^{t_n}(X) = X_1$
 2. $L(X_1) = l$

Let us imagine that we start the ODE at X_1 , and follow the flow $\Phi^T(X_1)$ for some time $T > 0$. On the one hand, we have

$$\Phi^T(X_1) = \Phi^T \left(\lim_{n \rightarrow +\infty} \Phi^{t_n}(X) \right) = \lim_{n \rightarrow +\infty} \Phi^{T+t_n}(X),$$

where we have used the continuity of the flow, and using the continuity of L we see that

$$L(\Phi^T(X_1)) = L\left(\lim_{n \rightarrow \infty} \Phi^{T+t_n}(X)\right) = \lim_{t \rightarrow \infty} L(\Phi^t(X)) = l,$$

so $L(\Phi^T(X_1)) = l$ but $L(X_1) = l$ and the quantity L is supposed to be strictly decreasing along the flow, which yields a contradiction.

□

3 Stability of linear systems

For homogeneous linear systems, there is always the zero solution (constant equal to 0), and it can be interesting to study its stability. In fact, by linearity, the stability of any solution is equivalent to the stability of the zero solution.

3.1 Constant coefficients

Let $X' = AX$ be an homogeneous linear ODE. Let $\lambda_1, \dots, \lambda_N$ be the eigenvalues of A .

Theorem 2. *We have the following conditions of (asymptotic) stability:*

- All the eigenvalues have a negative real part \iff asymptotic stability.
- All the eigenvalues have a non-positive real part and A diagonalizable \implies stability.

The proof follows from the explicit expression of the solutions that is available in this linear, constant coefficient case.

There is a subtlety concerning the existence of eigenvalues with 0 real part, if the algebraic multiplicity is not equal to the geometric multiplicity there is no stability. A simple example is given by

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

for which solutions are easy to compute, and we see that there is indeed no stability.

3.2 Non-autonomous case

There is no general result. The second exercise of HW9 shows that we may find variable coefficients $t \mapsto A(t)$ whose eigenvalues always have negative real part but such that zero is not a stable solution to $X' = A(t)X$.

In a perturbative setting, we have the following results: asymptotic stability persists under small, time-dependent perturbation of a constant coefficient.

Theorem 3. *Let A be a $N \times N$ matrix whose eigenvalues all have negative real parts. Let $t \mapsto P(t)$ be a continuous, matrix-valued function. If $\sup_t \|P(t)\|$ is small enough, depending on A , then the zero solution is still asymptotically stable for $X' = (A + P(t))X$.*

Proof. The proof is easy in the case $N = 1$, i.e. an ODE of the type

$$y' = -\lambda y + b(t)y,$$

where λ is positive. The solutions are given by

$$y(t) = e^{-\lambda t + \int_0^t b(s) ds} y(0),$$

and since $|\int_0^t b(s) ds| \leq t \sup_t |b(t)|$, we see that if $\sup_t |b(t)|$ is small enough (e.g. less than $\frac{1}{2}\lambda$), then $y(t)$ is bounded by $e^{-\frac{1}{2}\lambda t} y(0)$ which proves asymptotic stability.

I mentioned two wrong strategies of the general case... Here is a right one: We may write

$$Y(t) = e^{tA} Y(0) + \int_0^t e^{(t-s)A} P(s) Y(s) ds,$$

and we get, using the triangular inequality

$$\|Y(t)\| \leq \|e^{tA}\| \|Y(0)\| + \int_0^t \|e^{(t-s)A}\| \|P(s)\| \|Y(s)\| ds.$$

Now, using asymptotic stability of the unperturbed system, we know that all eigenvalues have a negative real part, and letting λ be one half of the real part which is the closest to zero, we have

$$\|e^{tA}\| \leq e^{t\lambda}$$

(perhaps up to a multiplicative constant). It yields, using the bound

$$\|Y(t)\| \leq e^{\lambda t} \|Y(0)\| + \sup_s \|P(s)\| \times \int_0^t e^{(t-s)\lambda} \|Y(s)\| ds.$$

We conclude using Grönwall's inequality in the integral form (see Wikipedia). □

Another result in the same spirit is.

Theorem 4. *Let A be a $N \times N$ matrix such that the zero solution is stable for $X' = AX$. Let $t \mapsto P(t)$ be a continuous, matrix-valued function. If $\int_0^{+\infty} \|P(s)\| ds$ is finite then the zero solution is still stable for $X' = (A + P(t))X$.*

Let us emphasize that Theorem 3 deals with asymptotic stability and Theorem 4 deals with stability.

4 Autonomous systems and their linearization

Theorem 5. *Let F be a C^1 map $\mathbb{R}^N \rightarrow \mathbb{R}^N$, with $F(0) = 0$, let A be the Jacobian of F at 0. Assume that the zero solution is asymptotically stable for $Y' = AY$. Then the zero solution is asymptotically stable for $Y' = F(Y)$.*

Proof. The proof goes in two steps.

First, we observe that the following function

$$L(x) := \int_0^{+\infty} \|e^{sA} x\|^2 ds,$$

is a Liapounov function for the equilibrium of $X' = AX$: it is easy to check that its only minimum is attained at 0, and decreasing along the flow of $X' = AX$, in fact we have

$$\frac{d}{dt} L(X(t)) = -\|X(t)\|^2$$

Let us note that here we **use the asymptotic stability** of the equilibrium to construct the Liapounov function, because we need to know something about the eigenvalues in order to guarantee that the integral defining $L(x)$ converges. So the spirit is completely opposite to that of Theorem 1. However, the point is that this Liapounov function, constructed for 0 as equilibrium of $X' = AX$, can still be used as a Liapounov function for 0 as equilibrium of $X' = F(X)$.

We want to show that L is decreasing under the flow of $X' = F(X)$. We introduce Φ^t the flow of this ODE and we compute

$$\frac{d}{dt}L(\Phi^t(X)) = \frac{d}{dt} \int_0^{+\infty} \|e^{sA}\Phi^t(X)\|^2 ds.$$

We obtain

$$\frac{d}{dt}L(\Phi^t(X)) = \int_0^{+\infty} \langle e^{sA}\Phi^t(X), e^{sA}F(\Phi^t(X)) \rangle ds.$$

A first-order approximation yields

$$F(\Phi^t(X)) = A\Phi^t(X) + \epsilon(\Phi^t(X)),$$

where $\lim_{x \rightarrow 0} \epsilon(x) = 0$. We obtain

$$\frac{d}{dt}L(\Phi^t(X)) = -\|\Phi^t(X)\|^2 + \epsilon(\Phi^t(X))\|\Phi^t(X)\|,$$

and if $\Phi^t(X)$ is small enough, this is bounded above by $-\frac{1}{2}\|\Phi^t(X)\|^2$. To summarize, if we start close enough to zero, then L decreases along the flow. □

Let us mention, to conclude, the Markus-Yamabe conjecture: if F is C^1 , with $F(0) = 0$ and is such that for all x in \mathbb{R}^N , the Jacobian matrix $DF(x)$ has all its eigenvalues with negative real parts, then the equilibrium solution at 0 is globally stable, in the sense that all solutions (no matter where we start from) converge to 0. This is true for $N = 2$ and false for $N > 2$.