1 Variational principle for the canonical Gibbs measure

Let $\mathsf{d} \geq 1$, let $\mathcal{P}([0,1]^{\mathsf{d}})$ denote the space of all probability measures on $[0,1]^{\mathsf{d}}$. For μ in $\mathcal{P}([0,1]^{\mathsf{d}})$, we define the *relative entropy* of μ with respect to the Lebesgue measure on $[0,1]^{\mathsf{d}}$ as

$$\mathsf{Ent}[\mu] := \int_{[0,1]^{\mathsf{d}}} \left(\frac{d\mu}{dx}\right) \log\left(\frac{d\mu}{dx}\right) dx,$$

if μ is absolutely continuous (with respect to the Lebesgue measure dx), and $+\infty$ otherwise.

Let W be a continuous function on $[0,1]^{\mathsf{d}}$. For any $\beta > 0$, we consider the *free energy functional* $\mathfrak{f}_\beta$ defined on $\mathcal{P}([0,1]^{\mathsf{d}})$ by

$$\mathfrak{f}_\beta(\mu) := \beta \mathbf{E}_\mu[W] + \mathsf{Ent}[\mu], \tag{1}$$

where $\mathbf{E}_\mu$ denotes the expectation under μ.

We want to prove the following *variational principle*: the unique minimiser of $\mathfrak{f}_\beta$ is the canonical Gibbs measure at (inverse) temperature β, whose density with respect to the Lebesgue measure is given by

$$\rho_\beta(x) = \frac{\exp(-\beta W(x))}{\int_{[0,1]^{\mathsf{d}}} \exp(-\beta W(x)) dx}$$

1. Argue that the *variational principle* amounts to minimising the following quantity among probability densities ρ.
$$\bar{\mathfrak{f}}_\beta(\rho) := \beta \int_{[0,1]^{\mathsf{d}}} W(x)\rho(x)dx + \int_{[0,1]^{\mathsf{d}}} \rho(x)\log\rho(x)dx.$$
2. Show that the derivative of $t \mapsto \bar{\mathfrak{f}}_\beta(\rho_\beta + t(\rho - \rho_\beta))$ at $t = 0$ vanishes for any probability density ρ.
3. Show that $\bar{\mathfrak{f}}_\beta$ is strictly convex, and conclude that ρ_β is its unique global minimiser.

2 Properties of the logarithmic energy

Let E be the space of compactly supported, continuous functions on $\mathbb{R}^2$, with mean 0.

1. Show that
$$D : (f,g) \mapsto \iint_{\mathbb{R}^2\times\mathbb{R}^2} -\log|x-y| f(x)g(y)$$
is a bilinear symmetric positive definite form on E.

   Hint for positivity: introduce the logarithmic potential $h^f(x) := \int -\log|x-y|f(y)dy$ associated to f and express $D(f,f)$ in terms of ∇h^f.
2. Let μ, ν be two probability measures on $\mathbb{R}^2$, with a continuous density with respect to the Lebesgue measure. Show that
$$2\iint_{\mathbb{R}^2\times\mathbb{R}^2} -\log|x-y| d\mu(x)d\nu(y)$$
$$\leq \iint_{\mathbb{R}^2\times\mathbb{R}^2} -\log|x-y| d\mu(x)d\mu(y) + \iint_{\mathbb{R}^2\times\mathbb{R}^2} -\log|x-y| d\nu(x)d\nu(y).$$
3. Deduce that the logarithmic energy functional (as in Section 2.1 of the lecture notes)
$$\mathcal{I}_V(\mu) := \iint_{\mathbb{R}^2\times\mathbb{R}^2} -\log|x-y| d\mu(x)d\mu(y) + \int_{\mathbb{R}^2} V(x)d\mu(x)$$
is strictly convex in μ.

3 Equilibrium measure

We refer to Section 2.1 in the lecture notes.

1. Let μ_{circ} be the *circular law* whose density is the uniform measure $\frac{1}{\pi}dx$ on the unit disk of $\mathbb{R}^2$.

   (a) Compute the logarithmic potential generated by μ_{circ}, i.e. compute the following quantity for any x in $\mathbb{R}^2$

   $$h^{\mu_{circ}}(x) := \int -\log|x-y|d\mu_{circ}(y).$$

   (b) Show that μ_{circ} satisfies the Euler-Lagrange equations for the quadratic potential $V(x) = |x|^2$, that is, prove that the quantity

   $$x \mapsto h^{\mu_{circ}}(x) + \frac{|x|^2}{2}$$

   is equal to a constant c on the unit disk and is larger than this constant outside the disk.

2. Show that the *arcsine law* of density $\frac{1}{\pi}\frac{1}{\sqrt{1-x^2}}$ on $[-1,1]$ is the equilibrium measure associated (in the `Log1` case) to the potential V that is constant on $[-1,1]$ and $+\infty$ outside.

3. (**) Let μ_{sc} be the *Wigner's semi-circular distribution* whose density is given by $\frac{1}{2\pi}\sqrt{4-x^2}$ on the line segment $[-2,2]$. Show that μ_{sc} satisfies the Euler-Lagrange equations for the quadratic potential $V(x) = x^2$ (in the `Log1` case).

4. Let μ be the equilibrium measure associated to a potential V (in the `Log1` case). We assume that V is C^1 and that μ has a smooth density with respect to the Lebesgue measure and is supported on a line segment.

   (a) Show that for any bounded continuous fonction h we have

   $$\iint \frac{h(x)-h(y)}{x-y}d\mu(x)d\mu(y) = \int V'(x)h(x)d\mu(x).$$

   (b) Show that, for x in the interior of the support of μ, we have

   $$2\int \frac{1}{x-y}d\mu(y) = V'(x),$$

   where the integral in the left-hand side is to be understood in the principal value sense.

4 Large deviation principles

We refer to Definition 2.5 in the lecture notes.

1. Let $\{x_N\}_N$ be a sequence of independent random variables on a space X, and P_N be the law of x_N. Assume that $\{P_N\}_N$ satisfies a LDP at speed N with rate function I, and that I has a unique minimiser x_{min} on X. Show that almost surely $\{x_N\}_N$ converges to $x_{\min}$ as $N \to \infty$, namely

   $$\forall \epsilon > 0, P(\liminf_{N\to\infty} |x_N - x_{min}| \leq \epsilon) = 1,$$

   where P is the product measure of the $\{P_N\}_N$'s.

2. Is the same result true for any speed?

3. (*) What is the asymptotic (as $N \to \infty$) macroscopic behavior of a system of N particles in the unit disk of $\mathbb{R}^2$ without interaction (i.e. their law is the Bernoulli point process with N points in the disk)? Is there almost sure convergence? A large deviation principle?