1 Joint law of the fluctuations for several test functions

Let us work in the two-dimensional Coulomb case, with a quadratic potential so that the associated equilibrium measure is the uniform measure on the unit disk. In this case, we have a central limit theorem for fluctuations of linear statistics as follows: If ξ is C^4, compactly supported inside the disk, then the quantity

$$\mathrm{Fluct}_N[\xi] := \sum_{i=1}^{N} \xi(x_i) - N \int_{D(0,1)} \xi(x) \frac{dx}{\pi}$$

converges to a Gaussian random variable with

$$\text{mean } = \frac{1}{2\pi} \left(\frac{1}{\beta} - \frac{1}{4} \right) \int_{\mathbb{R}^2} \Delta\xi \text{ and variance } = \frac{1}{2\pi\beta} \int_{\mathbb{R}^2} |\nabla \xi|^2.$$

Let ξ_1, ξ_2 be two smooth test functions supported in the unit disk.

1. Prove that any linear combination of the fluctuations of ξ_1 and ξ_2 (with real coefficients) converges to a Gaussian random variable, and specifiy the mean and variance.

2. Deduce that the vector $(\mathrm{Fluct}_N[\xi_1], \mathrm{Fluct}_N[\xi_2])$ converges to a Gaussian vector and specifiy the covariance.

   You may use the following result:

   Theorem 1 (Cramér-Wold) *Let X_n^1, X_n^2 and X^1, X_2 be random vectors. Then $\{(X_n^1, X_n^2)\}_n$ converges to (X_1, X_2) in distribution if and only if $\{(a_1 X_n^1 + a_2 X_n^2)\}_n$ converges to $a_1 X^1 + a_2 X^2$ for each a_1, a_2.*

3. Prove the Cramér-Wold theorem.

2 A large deviation principle

Let $\Lambda = [0,1]^{\mathsf{d}}$ and $g : \Lambda \times \Lambda \to [0, +\infty)$ by a symmetric function. Assume that g is continuous on $\Lambda \times \Lambda$. We let $W_N(\vec{X}_N)$ be defined as

$$W_N(\vec{X}_N) := \sum_{1 \le i \le j \le N} g(x_i, x_j).$$

We consider the Gibbs probability measure on Λ^N defined by

$$d\mathbf{P}_{N,\beta}^{\mathrm{cont}}(\vec{X}_N) := \frac{\exp\left(-\frac{\beta}{N} W_N(\vec{X}_N)\right)}{\mathbf{Z}_{N,\beta}^{\mathrm{cont}}} d\vec{X}_N,$$

with $\mathbf{Z}_{N,\beta}^{\mathrm{cont}}$ the normalization factor

$$\mathbf{Z}_{N,\beta}^{\mathrm{cont}} := \int_{\Lambda^N} \exp\left(-\frac{\beta}{N} W_N(\vec{X}_N)\right) d\vec{X}_N.$$

For a given $\vec{X}_N$, we let $\mu_N[\vec{X}_N] := \frac{1}{N} \sum_{i=1}^{N} \delta_{x_i}$ be the empirical measure.

1. Let μ be a probability measure on Λ. For $\epsilon > 0$, we let $B(\mu, \epsilon)$ be a ball around μ for some metric compatible with weak convergence of probability measures (it is not very important here). Show that

$$\limsup_{\epsilon \to 0} \limsup_{N \to \infty} \frac{1}{N} \log \mathbf{Z}^{\mathrm{cont}}_{N,\beta} \mathbf{P}^{\mathrm{cont}}_{N,\beta} \left(\left\{ \mu_N[\vec{X}_N] \in B(\mu, \epsilon) \right\} \right) \\ \leq -\beta \iint g(x,y) d\mu(x) d\mu(y) + \limsup_{\epsilon \to 0} \limsup_{N \to \infty} \frac{1}{N} \log \int_{\Lambda^N \cap \mu_N[\vec{X}_N] \in B(\mu,\epsilon)} d\vec{X}_N, \quad (1)$$

and prove a converse inequality with $\liminf$ instead of $\limsup$.

2. We recall the following result (Sanov's theorem).

Theorem 2 (Sanov)

$$\lim_{\epsilon \to 0} \lim_{N \to \infty} \frac{1}{N} \log \int_{\Lambda^N \cap \mu_N[\vec{X}_N] \in B(\mu,\epsilon)} d\vec{X}_N = -\mathsf{Ent}[\mu | \mathbf{Leb}] \quad (2)$$

Deduce that

$$\lim_{\epsilon \to 0} \lim_{N \to \infty} \frac{1}{N} \log \mathbf{Z}^{\mathrm{cont}}_{N,\beta} \mathbf{P}^{\mathrm{cont}}_{N,\beta} \left(\left\{ \mu_N[\vec{X}_N] \in B(\mu, \epsilon) \right\} \right) = -\beta \iint g(x,y) d\mu(x) d\mu(y) - \mathsf{Ent}[\mu | \mathbf{Leb}] \quad (3)$$

3. Using a finite covering by small balls $B(\mu, \epsilon)$ of the space of probability measures on Λ, show that

$$\lim_{N \to \infty} \frac{1}{N} \log \mathbf{Z}^{\mathrm{cont}}_{N,\beta} = - \inf_{\mathcal{P}(\Lambda)} \left(\beta \iint g(x,y) d\mu(x) d\mu(y) + \mathsf{Ent}[\mu | \mathbf{Leb}] \right),$$

and deduce a large deviation principle for the law of $\mu_N[\vec{X}_N]$.